Reto Spöhel, Rob van Stee **Summer 2012**

Exercises for Optimization

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss12/OPT

Exercise sheet 13 Due: **Tuesday, July 24, 2012**

You needed to collect at least 50% of all points on exercise sheets 1–6, and at least 50% of all points on exercise sheets 7–12. If you did not quite reach these goals, you can make up for missing points with this thirteenth exercise sheet.

Exercise 1 (*20 points*)

The quality of the lower bound given by a linear programming relaxation is usually measured as the ratio between Z_{IP}, the optimal value of the original integer programming problem, and Z_{LP}, the optimal value of its LP relaxation. This ratio is known as the *integrality gap* of the given integer programming problem.

Usually one is interested in bounding the integrality gap of *classes* of IP problems. E.g., a long-standing open question is: *What is the largest possible integrality gap that may arise in the subtour IP formulation (see next page) of* any *symmetric metric TSP instance?* ('Metric' means that the edge weights satisfy the triangle inequality.) It is known that this value lies between 4/3 and 3/2. In particular, solving the LP relaxation of the subtour IP always gives a lower bound that is off by a factor of at most 3/2.

We will discuss the upper bound of 3/2 in the lecture; this exercise concerns the lower bound of 4/3. Consider the (unweighted) graph $G' = (V, E')$ in Figure 1, and define a metric TSP instance $G = (V, E)$, $c : E \to \mathbb{N}_0$ as follows: $E := \binom{V}{2}$ (i.e. G is complete), and for $u, v \in V$, $u \neq v$, the weight c_e of the edge $e = \{u, v\}$ is given by the length (number of edges) of a shortest path from u to v in G'.

a) Show that the weight function c indeed satisfies the triangle inequality.

b) Show that the length of a shortest salesman tour in G (i.e. Z_{IP}) is $(4 + o(1))\ell$ as $\ell \to \infty$. *Hint:* Think of a tour in G as a closed walk in G' that visits all vertices.

c) Show that the value of the optimal solution of the linear programming relaxation of the subtour IP formulation (i.e. Z_{LP}) is $(3 + o(1))\ell$ as $\ell \to \infty$.

please turn over

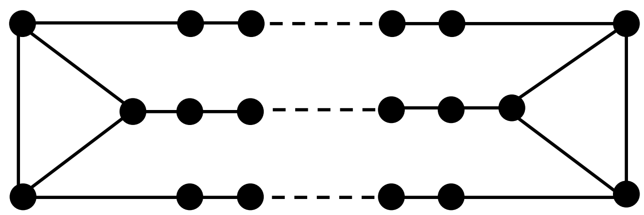

Figure 1: The graph G'. The dashed lines represent long paths; the three horizontal paths each have length ℓ in total.

The subtour IP formulation of TSP is:

$$\begin{aligned} \text{maximize} \quad & \sum_{e\in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e\in\delta(v)} x_e = 2, && v \in V, \\ & \sum_{e\in E(S)} x_e \leq |S| - 1, && \emptyset \neq S \subsetneq V, \\ & x_e \in \{0,1\}, && e \in E, \end{aligned}$$

where $\delta(v)$ denotes the set of edges incident to v, and $E(S)$ denotes the set of edges with both endpoints in S.

The LP relaxation is obtained by replacing the last constraint with

$$0 \leq x_e \leq 1, \qquad e \in E.$$

Exercise 2 (*12 points*)

A proper vertex-coloring of a graph G is a coloring of the vertices of G in which no two adjacent vertices receive the same color.

a) For a given graph G and an integer $k \geq 2$, give an integer program that is feasible if and only if G allows a proper vertex-coloring using at most k colors. The size of your program should be polynomial in the size of G.

b) Consider now the LP relaxation of your integer program. Give an example of a graph G and an integer k for which the LP relaxation is feasible but the original integer program is not.