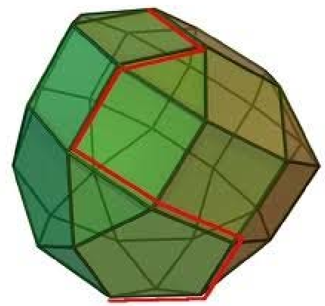
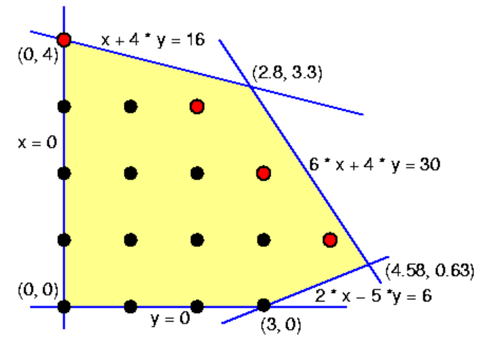
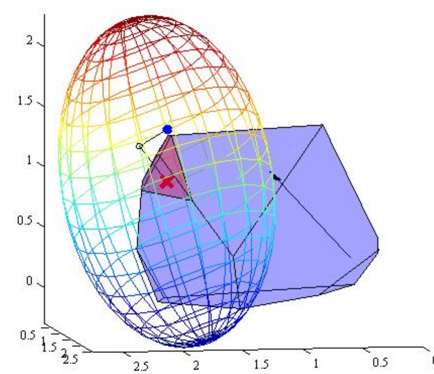
Optimization

- **Topic:** Fundamental concepts and algorithmic methods for solving **linear** and **integer linear programs**:
  - Simplex method
  - LP duality
  - Ellipsoid method
  - …





- **Prerequisites:** Basic math and theory courses
- **Credit:** 4+2 hours => 9 CP; written end of term exam
- **Lecturers:** Dr. Reto Spöhel, PD Dr. Rob van Stee
- **Tutors:** Karl Bringmann, Ruben Becker



- **Lectures:** Tue 10 –12 & Thu 12 –14 in E1.4 (MPI-INF), room 0.24
  Exercises: Wed 10 –12 / 14 –16

Optimization

Given:

- a **target function** $f(x_1, \ldots, x_n)$ of decision variables $x_1, \ldots, x_n \in \mathbb{R}$
  - e.g. $f(x_1, \ldots, x_n) = x_1 + 5x_3 - 2x_7$
  - or $f(x_1, \ldots, x_n) = x_1 \cdot (x_2 + x_3) / x_8$
- a **set of constraints** the variables $x_1, \ldots, x_n$ need to satisfy
  - e.g. $x_1 \geq 0$
    and $x_2 \cdot x_3 \leq x_4$
    and $x_4 + x_5 = x_6$
    and $x_1, x_3, x_5 \in \mathbb{Z}$

Geometric viewpoint: The set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ that satisfy all constraints is called the **feasible region** of the optimization problem.

Goal:

- find $x_1, \ldots, x_n \in \mathbb{R}$ **minimizing f** subject to the given constraints

I.e., find $(x_1, \ldots, x_n)$ **inside the feasible region** that minimizes f among all such points

Optimization

- Important special cases:
  - **convex optimization:** Both the feasible region and the target function are **convex**

    ➔ any local minimum is a global minimum (!)
  - $\supseteq$ **Semi-Definite Programming:** constraints are **quadratic** and semidefinite; target function is **linear**
  - $\supseteq$ **Quadratic Programming**: constraints are **linear**, target function is **quadratic** and semidefinite
  - $\supseteq$ **Linear Programming:** constraints are **linear**, target function is **linear**
  - **Integer [Linear] Programming:** Linear programming with additional constraint that some or all decision variable must be **integral** (or even $\in \{0,1\}$)

This course is mostly about LP and IP!

Linear Programming – A First Example

[Example taken from lecture notes by Carl W. Lee, U. of Kentucky]

- A company manufactures **gadgets** and **gewgaws**
- One kg of **gadgets**
  - requires **1 hour** of work, **1 unit** of wood, and **2 units** of metal
  - yields a **net profit of $5**
- One kg of **gewgaws**
  - requires **2 hours** of work, **1 unit** of wood, and **1 unit** of metal
  - yields a **net profit of $4**
- The company has **120 hours** of work, **70 units** of wood, and **100 units** of metal available
- What should it produce from these resources to **maximize its profit**?

Linear Programming – A First Example

- One kg of **gadgets**
  - requires **1 hour** of work, **1 unit** of wood, and **2 units** of metal
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- One kg of **gewgaws**
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x_1 := amount of gadgets

x_2 := amount of gewgaws

→

$$\begin{aligned} \max z &= 5x_1 + 4x_2 && \text{maximize profit} \\ \text{s.t. } x_1 + 2x_2 &\leq 120 && \text{work hours constraint} \\ x_1 + x_2 &\leq 70 && \text{wood constraint} \\ 2x_1 + x_2 &\leq 100 && \text{metal constraint} \\ x_1, x_2 &\geq 0 \end{aligned}$$

Linear Programming – A First Example

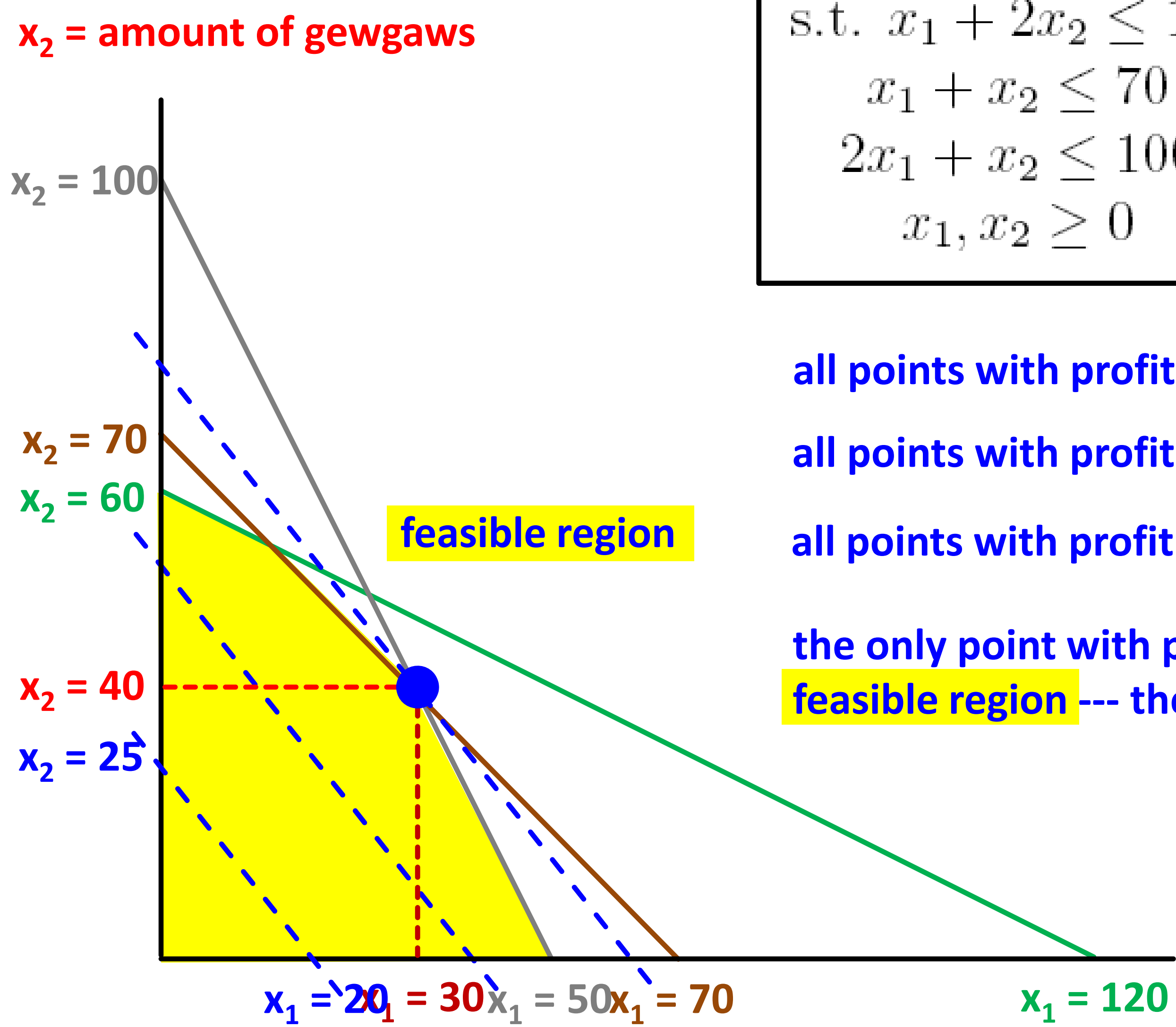
$$\max z = 5x_1 + 4x_2$$ maximize profit

$$\text{s.t. } x_1 + 2x_2 \leq 120$$ work hours constraint

$$x_1 + x_2 \leq 70$$ wood constraint

$$2x_1 + x_2 \leq 100$$ metal constraint

$$x_1, x_2 \geq 0$$



all points with profit $5x_1 + 4x_2 = 100$?

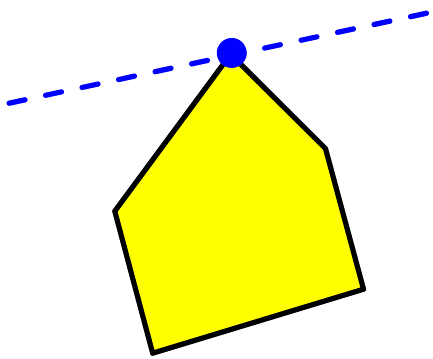
all points with profit $5x_1 + 4x_2 = 200$

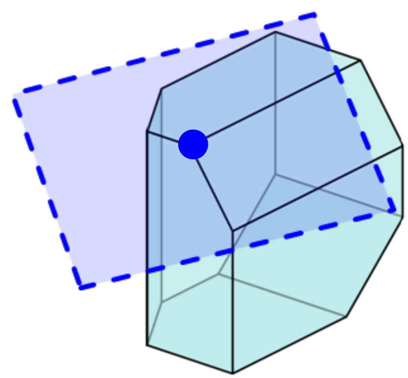
all points with profit $5x_1 + 4x_2 = 310$

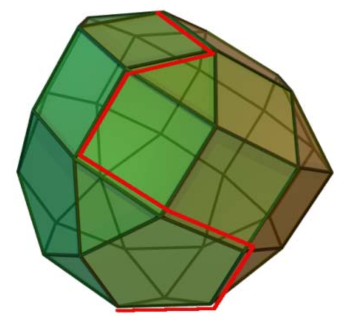
the only point with profit 310 inside the feasible region --- the optimum!

Observations / Intuitions

- It seems that in Linear Programming
  - the **feasible region** is always a **convex polygon/polytope**
  - the **optimum** is always attained at one of the **corners** of this polygon/polytope







- These intuitions are more or less true!
- In the coming weeks, we will **make them precise**, and exploit them to derive **algorithms for solving linear programs**.
  - First and foremost: **The simplex method**

LPs in Theoretical CS

- Many other CS problems can be cast as LPs.
- **Example:** MAXFLOW can be written as the following LP:

$$\begin{aligned}
&\textbf{maximize} \quad \sum_{v:(s,v)\in E} f(s,v) \\
&\textbf{subject to} \\
&\qquad 0 \le f(u,v) \le c(u,v) \qquad \text{for all } (u,v)\in E \\
&\text{and} \\
&\qquad \sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w) \qquad \text{for all } v \in V \setminus \{s,t\}
\end{aligned}$$

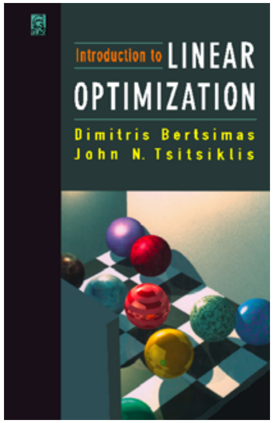
  – This LP has 2|E| + |V|-2 many constraints and |E| many decision variables.

- LPs can be solved in time **polynomial** in the input size!
- But the algorithm most commonly used for **solving them in practice** (simplex algorithm) is **not a polynomial algorithm!**

Books

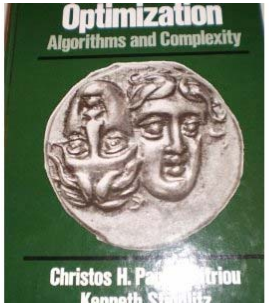
- Main reference for the course:

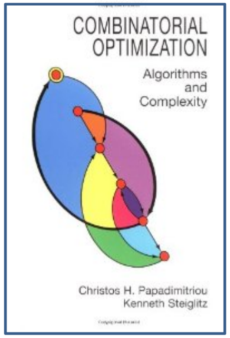
  Introduction to Linear Optimization
  by Dimitris Bertsimas and John N. Tsitsiklis
  - rather expensive to buy ☹
  - the library has ca. 20 copies
  - PDF of Chapters 1-5: google „leen stougie linear programming“



- Secondary reference:

  Combinatorial Optimization: Algorithms and Complexity
  by Christos H. Papadimitriou and Kenneth Steiglitz
  - very inexpensive (ca. €15 on amazon.de) ☺
  - ~~old~~ (1982) but covers most of the basics
    classic!





Exercises

- Exercise sessions start **in week 3** of the semester
- Two groups:
  - Wed 10-12 in E1.7 (cluster building), room 001
    Tutor: Ruben
  - Wed 14-16 in E 1.4 (this building), room 023
    Tutor: Karl

- Course registration and assignment to groups next week!

- Exercise sheets
  - will be put online Tuesday evening each week
  - should be handed in the following Tuesday in the lecture
  - You need to achieve 50% of all available points in the first and in the second half of the term to be admitted to the exam.

That‘s it for today.

On Thursday we will start with the definitions and theorems.