Solutions of Problem Set 1

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Problem 1. A DNF (disjunctive normal form) formula over boolean variables $x_1, \cdots, x_n$ is defined to be a logical OR of terms, each of which is a logical AND of literals (x_i or $\neg x_i$). Given a DNF formula φ and an integer k, we ask if it is possible to delete at least k terms so that the remaining formula is equivalent to φ. Show that this problem is in Σ_2.

Sample Solution. We first show that, for given formulae φ and φ' of n variables, the problem of testing if $\varphi \not\equiv \varphi'$ is in **NP**. To see this, note that we only need an assignment of these n variables as a certificate to show $\varphi \not\equiv \varphi'$, and this testing can be done in polynomial-time.

Now we prove that the original problem, called $\mathcal{P}$, is in Σ_2. For any given formula φ and integer k, a set of k' terms $D_1, \ldots, D_{k'}$ constitutes a certificate of $\mathcal{P}$. Given this certificate, we check: (i) whether these k' terms are different and $k' \geq k$; (ii) whether every D_i appears in φ; (iii) If the answer to questions (i) and (ii) is yes, then we test if $\varphi \not\equiv \varphi'$, where φ' is the formula formed by deleting k' terms $D_1, \ldots, D_{k'}$ of φ.

Since step (iii) can be done in **NP**, then original problem $\mathcal{P}$ is in Σ_2. ∎

Problem 2. Let X be a random variable. Show that for any deterministic function f it holds that $\mathbf{H}(f(X)) \leq \mathbf{H}(X)$.

Sample Solution. By definition, we have

$$\mathbf{H}(f(X)) = -\sum_{x \in X} \mathbf{Pr}\left[\, X = x \,\right] \cdot \log\left(\mathbf{Pr}\left[\, X = x \,\right]\right).$$

Hence

$$\begin{aligned}
\mathbf{H}(f(X)) &= -\sum_{y \in f(X)} \mathbf{Pr}\left[\, f(X) = y \,\right] \cdot \log \mathbf{Pr}\left[\, f(X) = y \,\right] \\
&= -\sum_{y \in f(X)} \left(\sum_{x \in f^{-1}(y)} \mathbf{Pr}\left[\, X = x \,\right] \right) \cdot \log \left(\sum_{x \in f^{-1}(y)} \mathbf{Pr}\left[\, X = x \,\right] \right) \\
&\leq -\sum_{y \in f(X)} \left(\sum_{x \in f^{-1}(y)} \mathbf{Pr}\left[\, X = x \,\right] \right) \cdot \log \left(\max_{x : x \in f^{-1}(y)} \{\mathbf{Pr}\left[\, X = x \,\right]\} \right) \\
&= -\sum_{y \in f(X)} \sum_{x \in f^{-1}(y)} \mathbf{Pr}\left[\, X = x \,\right] \cdot \log\left(\mathbf{Pr}\left[\, X = x \,\right]\right) \\
&= -\sum_{x \in X} \mathbf{Pr}\left[\, X = x \,\right] \cdot \log\left(\mathbf{Pr}\left[\, X = x \,\right]\right) \\
&= \mathbf{H}(X).
\end{aligned}$$

∎

Problem 3. For every $n, k, m \in \mathbb{N}$, every $\varepsilon > 0$ and every flat k-source X, let Ext be a function chosen randomly from

$$\mathcal{H} \triangleq \{f | f : \{0,1\}^n \mapsto \{0,1\}^m\}$$

where $m = k - 2\log(1/\varepsilon) - O(1)$. Show that $\mathsf{Ext}(X)$ is ε-close to $\mathcal{U}_m$ with probability $1 - 2^{-\Omega(K\varepsilon^2)}$, where $K = 2^k$ and $\mathcal{U}_m$ is the uniform distribution over $\{0,1\}^m$.

Sample Solution. Pick a function Ext randomly from $\mathcal{H}$. By the definition of ε-closeness, $\mathsf{Ext}(X)$ is ε-close to $\mathcal{U}_m$ if for any T, it holds that

$$|\mathbf{Pr}\,[\,\mathsf{Ext}(X) \in T\,] - \mathbf{Pr}\,[\mathcal{U}_m \in T\,]| \leq \varepsilon.$$

Note that x is called a flat k-source if X has a uniform distribution on $S \subseteq \{0,1\}^n$ with $|S| = 2^k$. Since X is flat k-source, we have

$$\mathbf{Pr}\,[\,\mathsf{Ext}(X) \in T\,] = \frac{|\{x \in \mathrm{Supp}(X) : \mathsf{Ext}(x) \in T\}|}{K}.$$

Also note that $\mathbf{Pr}\,[\mathcal{U}_m \in T\,] = \mu(T)$, where the density of set T is defined by $\mu(T) \triangleq |T|/2^m$. Since for every $x \in \mathrm{Supp}(X)$, the probability that $\mathsf{Ext}(x) \in T$ is $\mu(T)$, and these events are independent. By the Chernoff bound, for each fixed T, this condition holds with probability at least $1 - 2^{-\Omega(K\varepsilon^2)}$. Since there are 2^{2^m} different such T, the probability that the condition is violated for at least one T is at most $2^M 2^{-\Omega(K\varepsilon^2)}$, which is $2^{-\Omega(K\varepsilon^2)}$ for $m = k - 2\log(1/\varepsilon) - O(1)$. ∎

Problem 4. Suppose the feasible set of the LP

$$\begin{array}{ll} \text{maximize} & \mathbf{b}^{\mathrm{T}}\mathbf{z} \\ \text{subject to} & \mathbf{A}^{\mathrm{T}}\mathbf{z} \leq \mathbf{c} \end{array}$$

is nonempty and bounded, with $\|\mathbf{z}\|_\infty < \mu$ for all feasible $\mathbf{z}$. Show that any optimal solution of the problem

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^{\mathrm{T}}\mathbf{x} + \mu\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1 \\ \text{subject to} & \mathbf{x} \geq 0 \end{array}$$

is also an optimal solution of the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0. \end{array}$$

Sample Solution. Let L1, L2 and L3 be the three programs above. We have the following observations:

1. L3 is the dual of L1.
2. Since L1 is feasible and bounded, L3 is also feasible and bounded by the strong duality (Theorem 2 of the lecture notes). Moreover, L1 and L3 have the same optimal value.
3. An optimal solution of L3 is also an optimal solution of LP2.

Let $\mathbf{z}^\star$ and $\mathbf{x}^\star$ be optimal solutions of LP1 and LP3, respectively. By Observation 2, we have $\mathbf{b}^{\mathrm{T}}\mathbf{z}^\star = \mathbf{c}^{\mathrm{T}}\mathbf{x}^\star$. We prove that any optimal solution of L2, called $\mathbf{y}^\star$, is an optimal solution of L3. The proof is by contradiction. Assume that $\mathbf{y}^\star$ is not the optimal solution of L3. Then $\mathbf{A}\mathbf{y}^\star \neq \mathbf{b}$, which implies that $\|\mathbf{A}\mathbf{y}^\star - \mathbf{b}\|_1 \neq 0$. We have that the optimal solution of L2 is

$$\begin{aligned}
\mathbf{c}^{\mathrm{T}}\mathbf{y}^\star + \mu\|\mathbf{A}\mathbf{y}^\star - \mathbf{b}\|_1 &\geq (\mathbf{A}^{\mathrm{T}}\mathbf{z}^\star)^{\mathrm{T}}\mathbf{y}^\star + \mu\|\mathbf{A}\mathbf{y}^\star - \mathbf{b}\|_1 \\
&= (\mathbf{z}^\star)^{\mathrm{T}}\mathbf{A}\mathbf{y}^\star + \mu\|\mathbf{A}\mathbf{y}^\star - \mathbf{b}\|_1 \\
&> (\mathbf{z}^\star)^{\mathrm{T}}\mathbf{A}\mathbf{y}^\star + \|\mathbf{z}^\star\|_\infty\|\mathbf{A}\mathbf{y}^\star - \mathbf{b}\|_1 \\
&\geq (\mathbf{z}^\star)^{\mathrm{T}}\mathbf{A}\mathbf{y}^\star + \sum_i \|\mathbf{z}^\star\|_\infty \cdot |(\mathbf{A}\mathbf{y}^\star - \mathbf{b})_i| \\
&\geq (\mathbf{z}^\star)^{\mathrm{T}}\mathbf{A}\mathbf{y}^\star + \sum_i |\mathbf{z}_i^\star\mathbf{b}_i - \mathbf{z}_i^*(\mathbf{A}\mathbf{y}^\star)_i| \\
&\geq (\mathbf{z}^\star)^{\mathrm{T}}\mathbf{A}\mathbf{y}^\star + \sum_i \mathbf{z}_i^\star\mathbf{b}_i - \sum_i \mathbf{z}_i^\star(\mathbf{A}\mathbf{y}^\star)_i \\
&= (\mathbf{z}^\star)^{\mathrm{T}}\mathbf{A}\mathbf{y}^\star + \mathbf{b}^{\mathrm{T}}\mathbf{z}^\star - (\mathbf{z}^\star)^{\mathrm{T}}\mathbf{A}\mathbf{y}^\star \\
&= \mathbf{b}^{\mathrm{T}}\mathbf{z}^\star,
\end{aligned}$$

which contradicts to Observation 3 and the assumption that $\mathbf{y}^\star$ is an optimal solution. ∎