By *random variables* or *discrete random variables* we mean random variables taking either finitely many values or countably infinite values.

1. Given a positive integer k, describe a non-negative random variable X such that
$$\mathbf{Pr}[X \geq k\mathbb{E}[X]] = \frac{1}{k}.$$

2. Let X be a non-negative integer-valued random variable such that $X \leq m$, and $\mathbb{E}[X] \geq 2m^{1-t\delta/2}$. Prove that
$$\mathbf{Pr}\left[X \geq m^{1-t\delta/2}\right] \geq m^{-t\delta/2}.$$

3. Let the random variable X be given by $X = \sum_{i=1}^{n} X_i$. Show that if $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j]$ for every pair of i and j with $1 \leq i < j \leq n$, then $Var[X] = \sum_{i=1}^{n} Var[X_i]$.

4. Give an example of a random variable with finite expectation, and unbounded variance.

5. Let a and b be chosen independently and randomly from $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, where n ids a prime. Suppose we generate t pseudo-random numbers from $\mathbb{Z}_n$ by choosing $r_i = a.i + b \mod n$, for $1 \leq i \leq t$. For any $\epsilon \in [0, 1]$, show that there is a choice of the witness set $W \subset \mathbb{Z}_n$, such that $|W| \geq \epsilon n$, and the probability that none of the r_i's lie in the set W is at least $(1-\epsilon)^2/4$.

6. (Chernoff Bounds: Upper Tail) Let X be the sum of n independent indicator random variables, each equal to 1 with probability p, and zero otherwise. Let μ denote $\mathbb{E}[X]$.
   (a) Apply the substitution $Y = e^{tX}$. Given $\delta > 0$, express the event $X > (1+\delta)\mu$ in terms of Y.
   (b) Obtain an upper bound on the expression obtained in (a), by applying Markov's inequality to Y.
   (c) Obtain an upper bound on the moment generating function of X, i.e. $\mathbb{E}[Y]$, in terms of n, t and p.
   (d) Substitute the bound obtained in (c), to the expression obtained in (b).
   (e) Differentiate the expression obtained in (e) w.r.t. t and optimize to get the tightest possible upper bound.

7. (Chernoff Bounds: Lower Tail) Redo the previous exercise, but with the event $X < (1-\delta)\mu$, to get an upper bound on its probability of occurence.

8. Let $\mathcal{G}_p(n)$ be the random graph model having vertices $V = 1, 2, \ldots, n$, and each pair of vertices joined by an edge with probability $p = p(n)$ independently of the others.
   (a) The degree of a vertex $v \in V$ is the number of edges incident on v. Compute the expected degree of a vertex in $\mathcal{G}_p(n)$ in terms of n, p.
   (b) Let $p = n^{-\epsilon}$, where $\epsilon > 0$. Find the maximum degree of the random graph $\mathcal{G}_p(n)$, with probability tending to 1 as $n \to \infty$.

9. Suppose we have n jobs to distribute among m processors. [Assume m divides n]. A job requires one unit of time with probability p, and $k > 1$ units of time with probability $1-p$. Use Chernoff bounds, to derive upper and lower bounds on the time required (with high probability) for all jobs to be completed, if we randomly assign n/m jobs to each processor. (Notice the indicator variables are not $0-1$ variables here!)