Computational Geometry and Geometric Computing
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Winter 2009/2010
discussion in week starting October 19th

Exercise 1

Motivation

We study basic geometric predicates.

Orientation

Let $p = (p_x, p_y)$, $q = (q_x, q_y)$ and $r = (r_x, r_y)$ be three points in the plane. In class we proved that

$$\det \begin{pmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{pmatrix}$$

is twice the signed area of the triangle spanned by the points p, q, and r. The purpose of this exercise is to give an alternative proof.
Figure **??** shows a yellow parallelogram P. Prove that its area is $b'c - bc'$ by expressing the area as the difference of the large square and the rectangles R and the triangles T_1 and T_2.
What is the connection to the orientation predicate?

Side of Circle

Let $p = (p_x, p_y)$, $q = (q_x, q_y)$, $r = (r_x, r_y)$, and $s = (s_x, s_y)$ be four points in the plane. The first three points define an oriented circle (oriented from p via q to r). We want to determine whether s lies to the left or the right of the circle. If the circle is oriented counterclockwise, left corresponds to inside and right corresponds to outside. The purpose of this exercise is show that the following formula computes the *side-of-circle* predicate.

$$\operatorname{sign}\det \begin{pmatrix} 1 & p_x & p_y & p_x^2 + p_y^2 \\ 1 & q_x & q_y & q_x^2 + q_y^2 \\ 1 & r_x & r_y & r_x^2 + r_y^2 \\ 1 & s_x & s_y & s_x^2 + s_y^2 \end{pmatrix}$$

1. Prove that the determinant is zero if the four points are collinear.

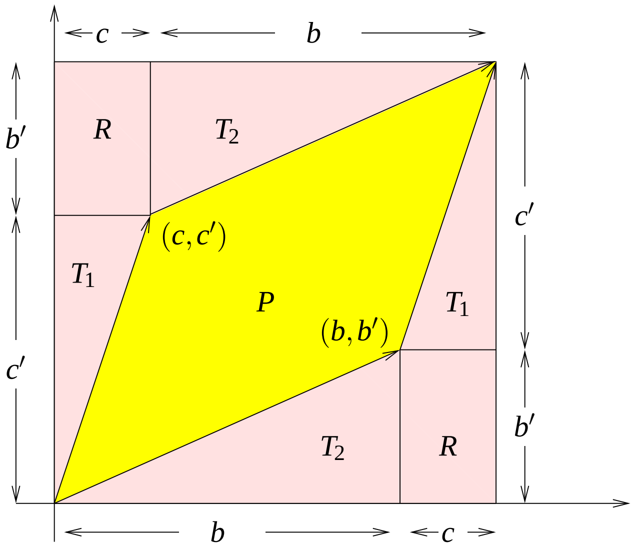

Figure 1: Area of parallelogram determined by vectors $(b, b'), (c, c')$.

2. From now on we assume that the four points are not collinear, say p, q and r define a proper triangle. For a point $v = (v_x, v_y)$ in the plane, let $\ell(v) = (v_x, v_y, v_x^2 + v_y^2)$ be the "lifting" of v to the paraboloid $z = x^2 + y^2$.

   (a) Assume first that the center of the circumcircle of $\Delta(p,q,r)$ is at the origin. Interpret the quantities in the fourth column.

   (b) Show that there are λ_p, λ_q, λ_r such that

   $$(1, s_x, s_y) = \lambda_p(1, p_x, p_y) + \lambda_q(1, q_x, q_y) + \lambda_r(1, r_x, r_y).$$

   Use this equation to simplify the last row of the determinant. What is the new diagonal entry in the last row?

   (c) Use the previous item to prove that the value of the determinant is equal to

   $$2 \cdot \text{signed-area}(\Delta(p,q,r)) \cdot (L^2 - R^2),$$

   where R is the radius of the circumcircle of Δ and L is the distance of s from the origin.

   (d) Conclude from the preceding item that the formula computes the side-of-circle predicate.

   (e) Generalize to the situation where the center of the circumcircle of Δ is not at the origin.

   (f) How should one interpret the formula if p, q, and r are collinear?

Have fun with the solution!