Computational Geometry and Geometric Computing
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Exercise 7

Motivation

We consider basic operations on polynomials such as root isolation and gcd computation.

GCD Computation

Let f be a polynomial with rational coefficients. Consider the following recursion: We initially start with $f_0 := f$ and $f_1 := f'$. For $i \geq 0$ let $d_i := \deg f_i - \deg f_{i+1}$ and consider

$$f_{i+2} := \lambda f_i + x^{d_i} f_{i+1}$$

with rational λ such that f_{i+2} has lower degree then f_i. Show:

- Let g be a rational polynomial that divides f and f', that is, there exists rational polynomials g_1 and g_2 with $f = g_1 \cdot g$ and $f' = g_2 \cdot g$. Then g divides each f_i.
- Let i_0 be the first index i where $f_i = 0$. Then f_{i_0-1} divides f and f' and there exists no polynomial of larger degree with the same property. It follows that $f_{i_0-1} = \gcd(f, f')$.
- If f is a rational polynomial with distinct complex roots $\xi_1, \ldots, \xi_m$ then

  $$f^* := (x - \xi_1) \cdot \ldots \cdot (x - \xi_m)$$

  is rational as well and a scalar multiple of $f/\gcd(f, f')$.
- $\deg \gcd(f, f') = \deg f - m$ with m as above.

Real Root Isolation

Given a polynomial $f = \sum_{i=0}^{n} a_i x^i$ with real coefficients we aim for a set of disjoint intervals $I_1, \ldots, I_m$ such that their union $\bigcup_{k=1}^{n} I_k$ contains all real roots of f and each I_k exactly one real root.

- Show that the modulus $|\xi|$ of each root ξ of f is bounded by

  $$B := 1 + \max_i \frac{|a_i|}{|a_n|}.$$

  (Hint: Each root ξ of f fulfills the inequality $|a_n||\xi|^n \leq \sum_{i=0}^{n} |a_i||\xi|^n$.)

- Let $I = (a, b)$ be an interval with midpoint $m = \frac{a+b}{2}$ and g a polynomial of degree N with Taylor expansion

$$g(m+x) = \sum_{k=0}^{N} \frac{g^{(k)}(m)}{k!} x^k$$

at m. We consider the test

$$T(g, I) : |g(m)| > \sum_{k=1}^{N} \frac{|g^{(k)}(m)|}{k!} \left(\frac{b-a}{2}\right)^k .$$

Show that I contains no root of f if $T(f, I)$ succeeds!

- Show: If $T(f', I)$ succeeds then f is monotone on I. How can you use this test to show that an interval I is isolating?

- Formulate an algorithm to isolate all real roots of f and show exactness and termination.

  (Hint: For the root isolation consider $f^* := f/\gcd(f, f')$ and use the fact that $(f^*)'(\xi) \neq 0$ at all roots ξ of f^*.)

Have fun with the solution!