Consider an undirected graph $G$. A random walk of length $\ell$ starting at vertex $u$ is a sequence of vertices $u = v_0, \ldots, v_\ell$, where each $v_i$ is chosen uniformly from the neighbors of $v_{i-1}$, for $i > 0$.

Traditionally, random walks were studied in infinite graphs. One of the basic results is as follows:

**Theorem 3.1 (Pólya, 1921)** Consider a random walk on an infinite $D$-dimensional grid. If $D = 2$, then with probability 1, the walk returns to the starting point an infinite number of times. If $D > 2$, then with probability 1, the walk returns to the starting point only a finite number of times.

In this lecture, we assume that $G$ is $d$-regular. Let $\pi_0$ be the initial distribution of vertices in $G$ and $\pi_i$ be the probability distribution of $v_i$ for a random walk beginning at vertex $v_0$. Then we observe the following facts.

- $\pi_1 = A\pi_0$.
- $\pi_{i+1} = A\pi_i = A(A\pi_{i-1}) = A^{i+1}\pi_0$.

**Definition 3.2** A distribution $\pi$ is called stationary if and only if $\pi = A\pi$.

For considering the convergence of $\{\pi_i\}$, we get the following result:

**Lemma 3.3** For every finite connected un-bipartite graph $G$, the distribution $\pi_i$ converges to a limit and stationary distribution. Moreover, if $G$ is regular, then this distribution is the uniform distribution $\mathbf{u}$ on $V$.

This lemma implies that, for any initial distribution $\pi_0$, if we take the random walk on an expander $G$, then after a finite number of steps, the probability that the random walk hits every vertex is uniform. However, for practical interests, we ask how fast $\pi_i$ converges to $\mathbf{u}$.

**Definition 3.4** For any vector $x$, define

$$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}.$$

In particular, let $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$.

**Definition 3.5 (Mixing Time)** The mixing time of a graph $G$ with $n$ vertices is the minimum $\ell$ such that for any staring distribution $\pi$,

$$\|A^\ell \pi - \mathbf{u}\|_\infty < \frac{1}{2n}.$$
A key property of random walks on an expander is that it converges rapidly to limit distributions.

**Theorem 3.6** If $G$ is a connected, $d$-regular, non-bipartite graph on $n$ vertices, then $\lambda < 1$ and $G$ has mixing time $O\left(\frac{\log n}{1-\lambda}\right)$.

**Proof:** Let $u = v_1, \ldots, v_n$ be the orthonormal eigenvalues of $A$. Then for any distribution $\pi$ on the vertices of $G$, we can write $\pi = \sum_{i=1}^{n} \pi_i$, where $\pi_i$ is a constant multiple of $v_i$. Therefore

$$
||A\pi - u||^2 = ||A\pi + A\pi_2 + \cdots + A\pi_n - u||^2
= ||\lambda_2\pi_2 + \lambda_3\pi_3 + \cdots + \lambda_n\pi_n||^2
= \lambda_2^2||\pi_2||^2 + \cdots + \lambda_n^2||\pi_n||^2
\leq \lambda^2(||\pi_2||^2 + \cdots + ||\pi_n||^2)
= \lambda^2(||\pi_2 + \cdots + \pi_n||^2)
= \lambda^2||\pi - u||^2
$$

which implies that $||A^\ell\pi - u|| \leq \lambda^\ell||\pi - u||$. Thus

$$
||A^\ell\pi - u|| \leq \lambda^\ell||\pi - u||
\leq \lambda^\ell||\pi||
\leq \lambda^\ell||\pi||_1
= \lambda^\ell
$$

and $||A^\ell\pi - u||_\infty < \frac{1}{2n}$ when $\ell = O\left(\frac{\log n}{\log \lambda}\right) = O\left(\frac{\log n}{1-\lambda}\right)$.

The above theorem also indicates that small $\lambda$ implies fast convergence of random walks.

Another important property of random walks is offered by the entropy of associated probability distribution. For any distribution $\pi$, the Shannon entropy of $\pi$ is defined by

$$
H(\pi) = -\sum_{i=1}^{n} \pi_i \log \pi_i.
$$

**Theorem 3.7** Let $A$ be the normalized adjacency matrix of an expander $G$. Then for any distribution $\pi$ on vertices of $G$, we have $H(A\pi) \geq H(\pi)$. 