max planck institut informatik

SAARLAND UNIVERSITY
COMPUTER SCIENCE

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Computer Algebra

https://resources.mpi-inf.mpg.de/departments/d1/teaching/ws14/ComputerAlgebra

Assignment sheet 11 due: Wednesday, January 28

Exercise 1: Descartes' Rule of Signs (4 points)

(a) Finalize the proof of Theorem IV.1.2., that is, show that if all roots of a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]$ are real, then $\text{var}(a_0, \ldots, a_n)$ equals the number of positive real roots of f.

(b) Prove the one-circle theorem, that is, show that $\text{var}(f, I) = 0$ if the one-circle region $A_0 \subset I$ of I (as defined in Theorem IV.1.4) contains no root of f.

Hint: You should first prove that $\text{var}(a_0, \ldots, a_n) = 0$ *if each root of* f *has a real part that is smaller than or equal to* 0*. Then, consider the image of the half plane* $H := \{x + i \cdot y \in \mathbb{C} : x \leq 0\}$ *under the Möbius transformation* Φ_I*.*

Exercise 2: Implementation of the Descartes method (4 points)

Implement the Descartes method and run your implementation on

- dense polynomials (i.e. most of the coefficients are non-zero) with randomly generated integer coefficients
- sparse polynomials (i.e. most of the coefficients are zero) with randomly generated integer coefficients
- polynomials of the form $x^n - (a \cdot x - 1)^2$, with varying degree n and a positive integer a of varying bit size.

What do you observe?

Exercise 3: Analysis of the Descartes method (4 points + 4 bonus points for (c))

(a) Derive a bound (of polynomial size in n and τ) on the number of iterations that is needed by the Descartes method to isolate the real roots of a polynomial $f \in \mathbb{Z}[x]$ of degree n with coefficients of bit size τ.

(b) Derive a bound on the number of bit operations that is needed in each iteration and provide a bit complexity bound for the overall algorithm that is polynomial in n and τ.

(c) Can you even give a bound of size $\tilde{O}(n\tau)$ in (a)? How does this affect the bit complexity of the Descartes method.

- *Hint: Consider the subdivision tree $\mathcal{T}$ induced by the Descartes method, where the nodes of $\mathcal{T}$ are the intervals produced by the algorithm. Notice that, for each non-terminal node I (i.e. $\operatorname{var}(f,I) \neq 0$), the one-circle region A_0 of I contains at least one root. Now, consider a mapping φ from the set of non-terminal nodes to the roots of f such that each root z of f has at most $O(\tau + \log \operatorname{sep}(z,f))$ pre-images. Finally, use Theorem II.3.7.*

Exercise 4: Newton-Rhapson Iteration (4 points)

Let α be a root of a polynomial $f \in \mathbb{R}[x]$. Provide a bound $\epsilon_0 > 0$ in terms of the degree n of f and the separation $\operatorname{sep}(\alpha, f)$ of α such that, for an arbitrary x_0 with $|x_0 - \alpha| < \epsilon_0$, the sequence

$$x_k := x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})} \quad \text{for } k \in \mathbb{N}_{>0}$$

converges against α under guarantee.

Hint: Use that $\frac{f'(x)}{f(x)} = \sum_{i=1}^{n} \frac{1}{x - z_i}$ for all x with $f(x) \neq 0$, where z_1 to z_n denote the roots of f.