Geometric Modeling 2010
Preparatory Exam

Wednesday, July 13th 2010

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General Instructions: Please read the following carefully before proceeding to solve the assignments!

- You have 2.5h time to finish the exam.
- You are allowed to bring three prepared hand written A4 sheets with you
- You can answer the questions in English or German.
- Please write **your full name** and your **student registration** number on **every sheet of paper** you use to write down solutions. Write the information in the upper right corner. Do this first before writing down any solutions. Sheets without this might possibly not be taken into account for grading.
- Please start a new page for each separate assignment.
- The assignments are roughly sorted by increasing difficulty, but this might vary subject to personal knowledge or preference.
- You can obtain 100 points. If you obtain at least 50 points you will pass the exam. Bonus points from the homework are not taken into account for passing, only to improve the grade in case of passing.
- **Good luck!**

Assignment #1: Misc. Questions **(4+3+3+3+4+3 = 20 points)**

a) Consider the following surfaces: A cylinder, a sphere, a cone and a planar disc. Which of these can be mapped isometrically to each other? More precisely this means: Is it possible to cut the source object into a finite number of pieces such that every piece can be mapped to a portion of the destination object using an isometric mapping function? Explain your answer briefly (2-3 sentences overall should suffice).

(1) cylinder (no caps) **(2) sphere** **(3) cone (no cap)** **(4) planar disc**

b) Given the function $f(x) = x^3 - x^2 - \frac{1}{4}x + 1$. Compute the curvature of the function graph at the point (1, f(1)).

c) Give an implicit signed distance function for the unit circle (center (0,0), radius 1, inside is negative, no justification necessary).

d) Assume we are given a cubic Bezier curve segment. We know that 3 control points are located at one and the same point and a fourth point is at a different position. How does the resulting curve (the curve traced out by the parametric object, neglecting the parametric trajectory) look like? Why?

e) A quadratic polynomial in Bézier form can be expressed in matrix notation as:

$$F(t)=[t^2\ t\ 1]\mathbf{B}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}$$

where p_0, p_1, p_2 are Bézier control points (this is the 1D case, so these are just real numbers) and t is the parameter of the polynomial. Write down the Bernstein basis for degree 2 and compute the matrix **B** that translates to the monomial basis.

f) Show that the Bernstein basis of degree d forms a partition of unity, i.e.: $\forall t: \sum_{i=0}^{d} b_i^{(d)}(t)=1$.

Assignment #2: Blossoms, Polars & de Casteljau **(5+5+5+15+10 = 40 points)**

a) Write down the polar forms of the following polynomials. The polar forms should be of degree 3:

$$F_1(t)=1$$
$$F_2(t)=t$$
$$F_3(t)=t^2$$
$$F_4(t)=2t+4t(1-t)+7-5t^3$$

b) Determine the Bezier control points (this is again the 1D case, so these are just real numbers) of a cubic Bezier curve coinciding with F_4. Give the control points for a parametrization of the Bezier segment over the interval $t \in [0,1]$.

c) The polar form f of degree 2 is defined implicitly by control points: $f(0,0)=1,\ f(0,1)=2,\ f(1,1)=0$

(1) Compute the polynomial in standard monomial representation (F(t) = at^2 + bt + c).

(2) Degree elevation: Compute the polar form of degree 3 that describes the same polynomial.

d) (Please read the remarks at the end of this assignment) A cubic Bezier curve in the plane is defined by the following control points:

$$\mathbf{p}_0=\begin{pmatrix}1\\1\end{pmatrix}, \mathbf{p}_1=\begin{pmatrix}2\\2\end{pmatrix}, \mathbf{p}_2=\begin{pmatrix}4\\2\end{pmatrix}, \mathbf{p}_3=\begin{pmatrix}5\\1\end{pmatrix}$$

(1) Draw the control points and the control polygon.

(2) Annotate the control points in blossom notation, assuming the Bezier curve starting at p_0 and ending at p_3 is parametrized over the interval $t \in [0,1]$.

(3) Perform the de Casteljau algorithm to compute the curve value at t = 0.5. Draw the intermediate steps in your drawing and write down in one line each how you have computed the intermediate points (for example: **f**(1,2,3) = 0.5·**f**(0,2,3) + 0.5·**f**(2,2,3)). Annotate all intermediate points in blossom notation in your drawing. Please also annotate your drawing with split ratios for the control polygon segments (see remarks).

(4) Assume you want to continue the curve in a C^1 continuous fashion with a second Bezier segment – what constraints do the four new control points have to meet? Give a proof for your answer.

e) (Please read the remarks at the end of this assignment) A cubic, uniform B-spline curve segment is defined by the control points

$$\mathbf{p}_0 = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{p}_1 = \begin{pmatrix}2.5\\4\end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix}4.5\\4\end{pmatrix}, \mathbf{p}_3 = \begin{pmatrix}6\\1\end{pmatrix}$$, parametrized over the interval $t \in [2..3]$.

(1) Draw the control polygon and annotate it with blossom values for the control points.

(2) Compute the Bézier control points that describe the same curve. Again, write down in one line each how you have computed the intermediate points. Annotate all intermediate points in blossom notation in your drawing. Please also annotate your drawing with split ratios for the control polygon segments (see remarks).

Remarks for Assignment 2d, 2e:

- We suggest to use a scale of at least 1 unit ≙ 2 cm on an A4 sheet (portrait, more for landscape).
- You can use a ruler to split the control polygons in the necessary ratios. You do *not need* to compute the numerical values for all of the constructed points as long as you *both* write down how you compute the blossom values and annotate the control polygon segments with numbers indicating the split ratios.

Assignment #3: Total least squares **(15+5 = 20 points)**

Given n points $\mathbf{p}_1,\dots,\mathbf{p}_k$ in three dimensional Euclidean space, a best fitting plane in a least-squares sense can be computed by PCA (principal component analysis). The procedure is the following:

- Compute the average $\mathbf{p}_{av} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{p}_i$. Subtract this from every point: $\mathbf{d}_i = \mathbf{p}_i - \mathbf{p}_{av}$
- Form the scatter matrix $\mathbf{S} = \mathbf{X}\mathbf{X}^T, \mathbf{X} := \begin{pmatrix} | & & | \\ \mathbf{d}_1 & \dots & \mathbf{d}_n \\ | & & | \end{pmatrix}$.
- The plane is defined by the average $\mathbf{p}_{av}$ as one point in the plane and the eigenvector of $\mathbf{S}$ with the smallest eigenvalue as the normal of the plane (we assume this vector to be uniquely determined up to sign and length).

An alternative approach to plane fitting is “plate tensor voting”. The procedure is the following:

- Compute the average as before.
- Compute the vector from the average to each point, denoted by $\mathbf{d}_i$. Let $l_i = \|\mathbf{d}_i\|$ be the length of the vector.
- Form the matrices $\mathbf{M}_i = l_i\mathbf{I} - [\mathbf{d}_i]\cdot[\mathbf{d}_i]^{\mathrm{T}}$.
- Average all the matrices $\mathbf{M}_i$: $\mathbf{M} := \frac{1}{n}\sum_{i=1}^{n}\mathbf{M}_i$.
- Compute the normal vector as eigenvector corresponding to the largest eigenvalue of $\mathbf{M}$. The plane is defined by this normal vector and the average point (same point as before).

a) Show formally that both schemes compute the exact same solution (i.e., the same plane). You can assume that all eigenvalues are different so that the plane is uniquely defined.

b) Explain in 1-2 sentences what the matrices $\mathbf{M}_i$ represent intuitively (hint: the name of the scheme is derived from this observation). Why do we expect them to sum up to a matrix for which the eigenvector with largest eigenvalue is the normal of the plane? (1-2 sentences are sufficient).

Assignment #4: Subdivision Curves **(10+10 = 20 points)**

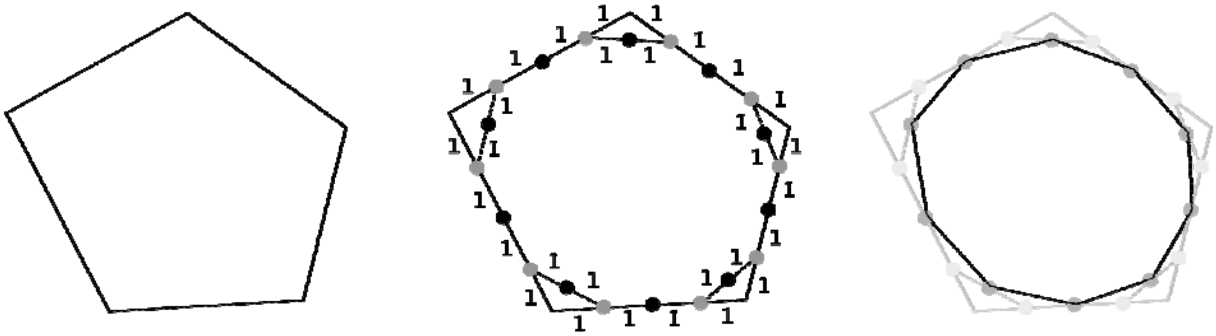

Figure: Dyn's subdivision scheme

Dyn's algorithm is a variant of corner cutting. In Dyn's algorithm, every segment of a closed control polygon is divided into four equally sized parts (1:1:1:1, see Figure above). Then, the last point of each segment is connected with the first point of the subsequent segment, and this connecting edge is divided in half (1:1, dashed line in the Figure). The refined control points are the mid-points of the original segments and the midpoints of the connecting edges. Show that Dyn's algorithm converges to a C^2-continuous, piecewise cubic curve. Do this in two steps:

a) Consider the corners of the control polygon as control points of a uniform cubic B-spline with knot sequence t = (...,0,1,2,3,4,5,...). Perform knot insertion to insert a new knot value at position t = 2.5. This divides the spline segment $t \in [2,3]$ into two new segments $t \in [2, 2.5]$, $t \in [2.5, 3]$. What are the control points of the new segments in blossom notation?

b) Insert another knot at position t=1.5. Make sure to start from your result in (a) so that you end up with a knot sequence t=(...,0,1,1.5,2,2.5,3,4,5,...). Compare the resulting control points to the ones generated by Dyn's algorithm. If you can show that they coincide, it follows from symmetry that one step of Dyn's algorithm is equivalent to subdividing each interval of the original knot sequence by knot insertion. Since repeated knot insertion lets the control polygon converge to the curve itself, this concludes the proof.