Geometric Modeling

Summer Semester 2010

Mathematical Tools (2)

Recap: Analysis · Differential Geometry

UNIVERSITÄT DES SAARLANDES

CLUSTER OF EXCELLENCE

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Today...

Topics:

- Mathematical Background
  - Linear algebra
  - Analysis & differential geometry
  - Numerical techniques

Overview

Analysis

- Multi-dimensional differentiation
- Multi-dimensional integration

Differential Geometry

- Length, area, volume
- Curvature of curves and surfaces

Recap: Analysis

Multi-Dimensional Differentiation

Derivative of a Function

Reminder: The *derivative* of a function is defined as

$$\frac{d}{dt}f(t) := \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

If this limit exists, the function is called differentiable (on the set, on which the limit converges).

Other notation:

$$\frac{d}{dt}f(t) = \underbrace{f'(t)}_{\substack{\text{in case the}\\ \text{reference is}\\ \text{clear from}\\ \text{the context}}} = \underbrace{\dot{f}(t)}_{\substack{\text{in case the}\\ \text{variable}\\ \text{is the time}}}$$

Repeated Differentiation

Multiple Differentiation:

- The differentiation operation can be applied repeatedly:

$$\frac{d^2}{dt^2} f(t) = f''(t) = \ddot{f}(t) = \lim_{h \to 0} \frac{f(t+h) - 2f(t) + f(t-h)}{h^2}$$

$$\frac{d^k}{dt^k} f(t) = f^{(k)}(t)$$

- A function that can be k-times differentiated resulting in a smooth function belongs to the set C^k.
  Abuse of notation: C^0 means just smoothness. C^{-1} means "not even smooth".

Taylor Approximation

Smooth functions can be approximated locally:

- $$\begin{aligned} f(x) \approx{}& f(x_0) \\ &+ \frac{d}{dx} f(x_0)(x - x_0) \\ &+ \frac{1}{2}\frac{d^2}{dx^2} f(x_0)(x - x_0)^2 + \ldots \\ &\ldots + \frac{1}{k!}\frac{d^k}{dx^k} f(x_0)(x - x_0)^k + O(x^{k+1}) \end{aligned}$$
- Guaranteed to converge globally for holomorphic functions
- Usually a good local approximation for smooth functions

Partial Derivative

If a function depends on more than one variable:

- The notation changes: (but the rest remains the same)

use curly-d

$$\frac{\partial}{\partial x_k} f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) :=$$

$$\lim_{h \to 0} \frac{f(x_1, \ldots, x_{k-1}, x_k + h, x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)}{h}$$

- Derivative in one direction $\mathbf{x}_k$ only, other variables remain constant.
- This is called a partial derivative. Other notation:

$$\frac{\partial}{\partial x_k} f(\mathbf{x}) = \partial_k f(\mathbf{x}) = f_{x_k}(\mathbf{x})$$

Special Cases

Derivatives for:

- Functions f: $\mathbb{R}^n \to \mathbb{R}$ (“heightfield”)
- Functions f: $\mathbb{R} \to \mathbb{R}^n$ (“curves”)
- Functions f: $\mathbb{R}^n \to \mathbb{R}^m$ (general case)

Special Cases

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Gradient

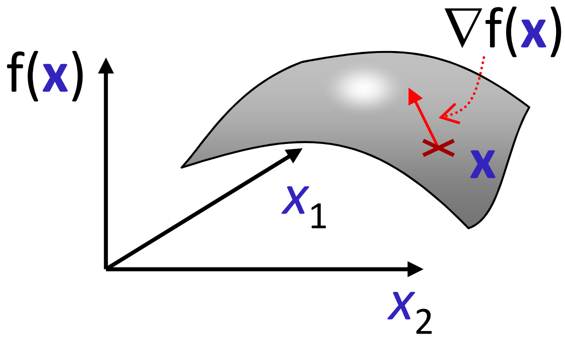
Gradient:

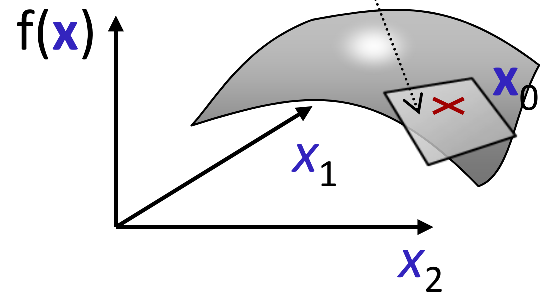
- Given a function f: $\mathbb{R}^n \to \mathbb{R}$ ("heightfield")
- The vector of all partial derivatives of f is called the *gradient*:

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} f(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_n} f(\mathbf{x}) \end{pmatrix}$$

Gradient

Gradient:





- The gradient is a vector that points in the direction of steepest ascent.
- Local linear approximation (first order Taylor approx.):

$$f(\mathbf{x}) \approx f(\mathbf{x}_0)+\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)$$

- These are all heightfields (f: $\mathbb{R}^n \to \mathbb{R}$)

Higher Order Derivatives

Higher order Derivatives:

- Can do all combinations: $\left(\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \ldots \frac{\partial}{\partial x_{i_k}}\right) f$
- Order does not matter for $f \in C^k$

Hessian Matrix

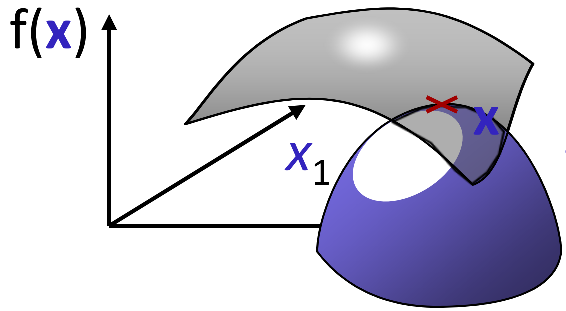
Higher order Derivatives:

- Important special case: Second order derivative

$$\begin{pmatrix} \frac{\partial^2}{\partial {x_1}^2} & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} & \frac{\partial^2}{\partial {x_1}^2} & & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_2} \\ \vdots & & \ddots & \vdots \\ \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_n} & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_n} & \cdots & \frac{\partial^2}{\partial {x_n}^2} \end{pmatrix} f(\mathbf{x}) =: H_f(\mathbf{x})$$

- "Hessian" matrix (symmetric for $f \in C^2$)

Taylor Approximation



2nd order approximation (schematic)

Second order Taylor approximation:

- Fit a paraboloid to a general function

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}} \cdot H_f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

Special Cases

Derivatives for:

- Functions f: $\mathbb{R}^n \to \mathbb{R}$ ("heightfield")
- Functions f: $\mathbb{R} \to \mathbb{R}^n$ ("curves")
- Functions f: $\mathbb{R}^n \to \mathbb{R}^m$ (general case)

Derivatives of Curves

Derivatives of vector valued functions:

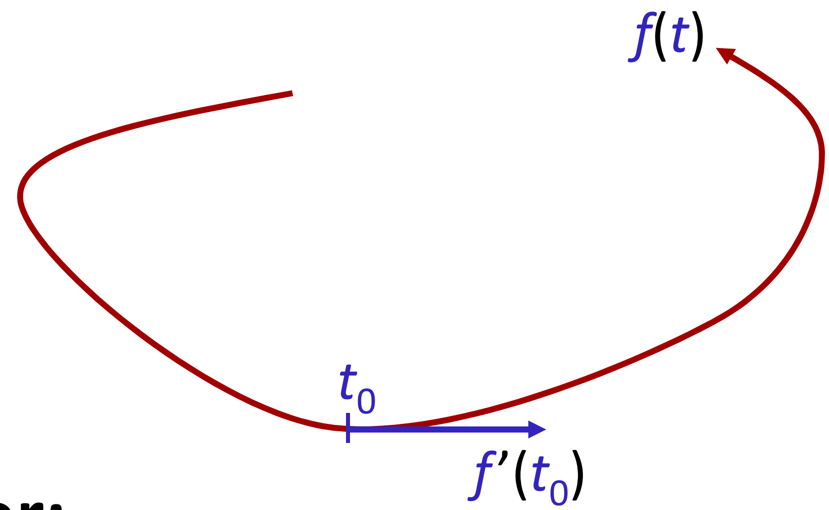
- Given a function f: $\mathbb{R} \to \mathbb{R}^n$ ("curve")

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

- We can compute derivatives for every output dimension:

$$\frac{d}{dt} f(t) := \begin{pmatrix} \frac{d}{dt} f_1(t) \\ \vdots \\ \frac{d}{dt} f_n(t) \end{pmatrix} =: f'(t) =: \dot{f}(t)$$

Geometric Meaning



Tangent Vector:

- f' is a vector in tangent direction
- If f describes the motion of a physical particle, $\dot{f}$ is its velocity vector.
- Higher order derivatives: Again vector functions
- The second derivative $\ddot{f}$ is the acceleration vector.

Special Cases

Derivatives for:

- Functions f: $\mathbb{R}^n \to \mathbb{R}$ (“heightfield”)
- Functions f: $\mathbb{R} \to \mathbb{R}^n$ (“curves”)
- Functions f: $\mathbb{R}^n \to \mathbb{R}^m$ (general case)

You can combine it...

General case:

- Given a function f: $\mathbb{R}^n \to \mathbb{R}^m$ ("space warp")

$$f(\mathbf{x}) = f\big((x_1, \ldots, x_n)\big) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{pmatrix}$$

- Maps points in space to other points in space
- Now we can compute derivatives of all *output components* of f in all *input directions*.
- This is called the Jacobian matrix, usually still denoted by ∇, like the gradient

Jacobian Matrix

Jacobian Matrix:

$$\nabla f(\mathbf{x}) = J_f(\mathbf{x}) = \nabla f(x_1,\ldots,x_n)$$
$$= \begin{pmatrix} \nabla f_1(x_1,\ldots,x_n)^{\mathrm{T}} \\ \vdots \\ \nabla f_m(x_1,\ldots,x_n)^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \partial_{x_1} f_1(\mathbf{x}) & \cdots & \partial_{x_n} f_1(\mathbf{x}) \\ \vdots & & \vdots \\ \partial_{x_1} f_m(\mathbf{x}) & \cdots & \partial_{x_n} f_m(\mathbf{x}) \end{pmatrix}$$

Use in a first-order Taylor approximation:

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + J_f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

matrix / vector
product

Recap: Analysis

Properties of Multi-Dimensional Derivatives

Coordinate Systems

Problem:

- What happens, if the coordinate system changes?
- Partial derivatives go into different directions then.
- Do we get the same result?

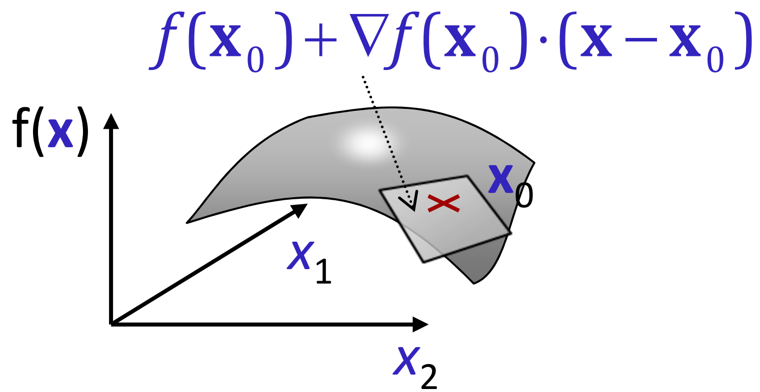
Total Derivative

First order Taylor approx.:

- $f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + R_{x_0}(\mathbf{x})$
- If this converges to the function with first order, i.e.

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{R_{x_0}(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0,$$

  the function is called *totally differentiable*.
- This means, the function can be locally well-approximated with a linear function.
- Every C[1] function f: $\mathbb{R}^n \to \mathbb{R}^m$ is totally differentiable



Partial Derivatives

Consequences:

- A linear function is fully determined by the image of a basis (a linear independent set)
- This means: The directions of partial derivatives do not matter – this is just a basis transform.
- We can use any linear independent set of directions **T** and transform to standard basis by multiplying with $\mathbf{T}^{-1}$.

Directional Derivative

The directional derivative is defined as:

- Given a function f: $\mathbb{R}^n \rightarrow \mathbb{R}^m$, and a direction $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq \mathbf{0}$.
- Then the directional derivative of f in direction $\mathbf{v}$ is given by:

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}) := \frac{d}{dt} f(\mathbf{x} + t\mathbf{v})$$

- In case of total differentiability, the directional derivative can be computed using the Jacobian matrix:

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Rules of Thumb

Rules of Thumb

- "∇" means differentiation (everything by everything)
  - Might be a scalar (1D$\rightarrow$1D),
  - a vector (kD$\rightarrow$1D, 1D$\rightarrow$kD), or
  - a matrix (nD$\rightarrow$mD).
- "Jacobian" means "∇"
- Higher order derivatives:
  - Can combine directions $\partial x\ \partial y\ \partial z$...
  - Order (usually) does not matter (continuity of deriv. sufficient)
  - Hessian matrix: 2nd order derivative (kD$\rightarrow$1D)

Rules of Thumb

More Rules of Thumb

- Coordinate system (usually) does not matter
- The derivative is just local linear approximation
  - We can compute this in different (linear independent) directions
  - Do a basis transform to change coordinates
  - To evaluate in a single direction: Multiply Jacobian matrix (first order derivative) with unit direction vector.

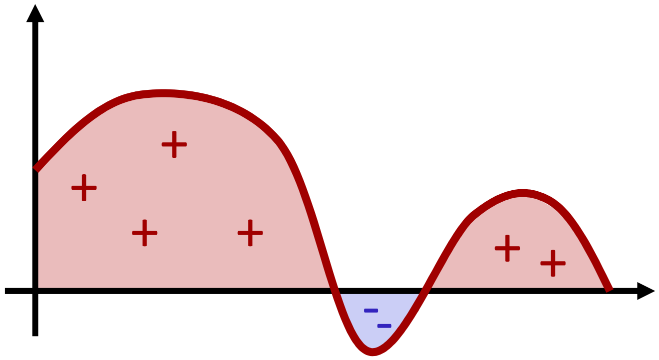
Recap: Analysis

Multi-Dimensional Integration

Integral

Integral of a function

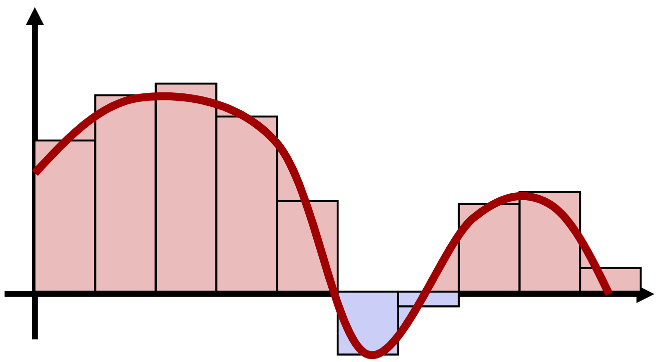
- Given a function $f\colon \mathbb{R} \to \mathbb{R}$
- The integral $\int_a^b f(t)dt$ measures the signed area under the curve:



Integral

Numerical Approximation

- Sum up a series of approximate shapes

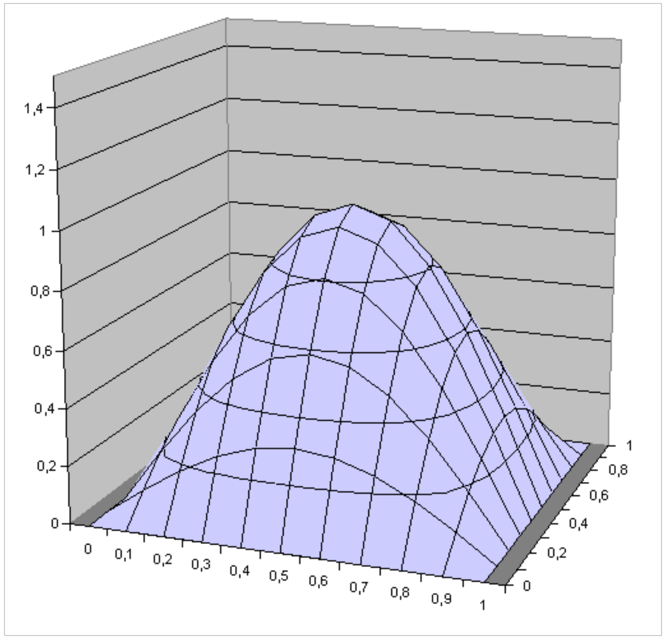


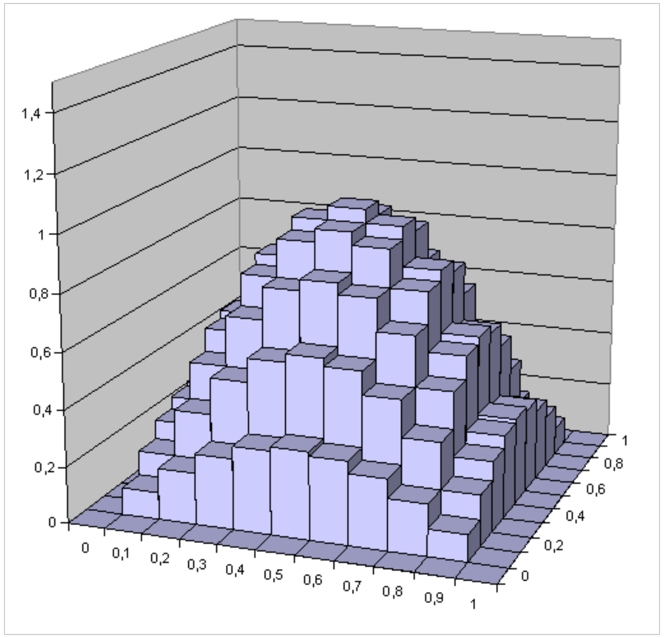
- (Riemannian) Definition: limit for baseline $\to$ zero

Multi-Dimensional Integral

Integration in higher dimensions

- Same idea.
- Consider functions f: $\mathbb{R}^n \to \mathbb{R}$
- Tessellate domain and sum up volume of cuboids



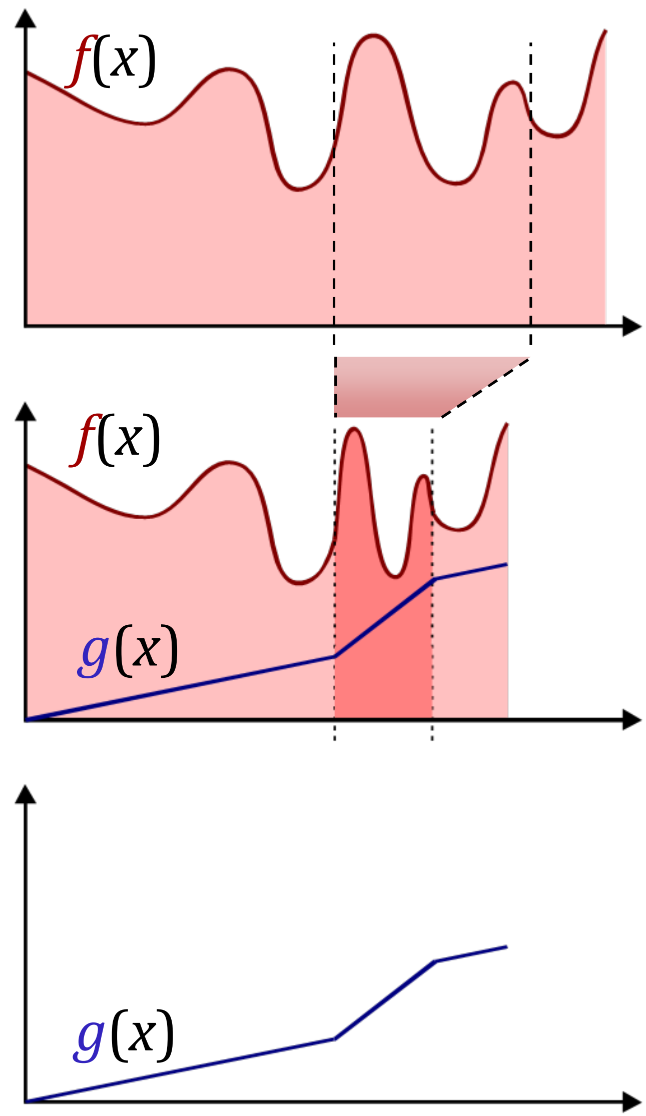


Integral Transformations

Integration by substitution:

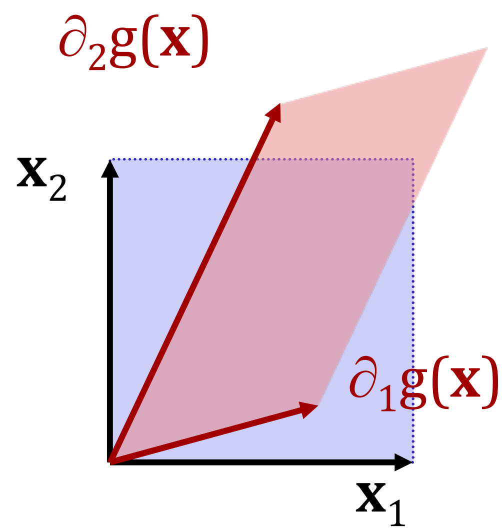
$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)dt$$

Need to compensate for speed of movement that shrinks the measured area.



Multi-Dimensional Substitution

Transformation of Integrals:



$$\int_{\Omega} f(\mathbf{x})d\mathbf{x} = \int_{g^{-1}(\Omega)} f\big(g(\mathbf{y})\big)\left|\det\big(\nabla g(\mathbf{y})\big)\right| d\mathbf{y}$$

- $g \in C^1$, invertible
- Jacobian approximates local behavior of *g*() (linear consistency sufficient)
- Determinant computes local area/volume change
- In particular: $\left|\det\big(\nabla g(\mathbf{y})\big)\right| = 1$ means *g*() is *area/volume conserving*.

Differential Geometry

of Curves & Surfaces (Overview)

Part I: Curves

Parametric Curves

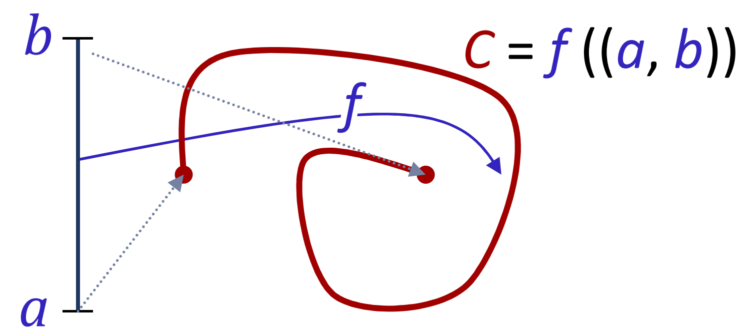
Parametric Curves:

- A differentiable function

  $f\colon (a, b) \to \mathbb{R}^n$

  describes a *parametric curve*

  $C = f((a, b)),\ C \subseteq \mathbb{R}^n$.



- The parametrization is called *regular* if $f'(t) \neq 0$ for all t.
- If $\|f'(t)\| \equiv 1$ for all t, f is called a *unit-speed parametrization* of the curve C.

Length of a Curve

The length of a curve:

- The length of a regular curve C is defined as:

$$\mathrm{length}(C) = \int_a^b \left\| f'(t) \right\| dt$$

- This definition is independent of the parametrization (integral transformation theorem).

- Alternatively, the length of the curve can be defined as length(C) = $|b - a|$ for a unit-speed parametrization $C = f((a, b))$; this obviously yields the same result.

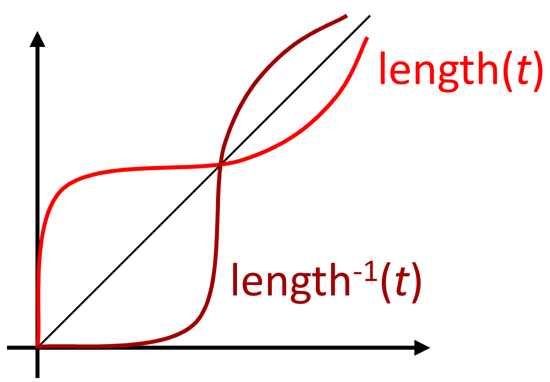
Reparametrization

Enforcing unit-speed parametrization:

- Assume: $\|f'(t)\| \neq 0$ for all t.
- We have:

$$\text{length}(C) = \int_a^b \|f'(t)\| dt$$ (invertible, because $f'(t) > 0$)

- Concatenating $f \circ \text{length}^{-1}(C)$ yields a unit-speed parametrization of the curve



Tangents

Unit Tangents:

- The unit tangent vector at $x \in (a, b)$ is given by:

$$\text{tangent}(t) = \frac{f'(t)}{\left\| f'(t) \right\|}$$

- For curves $C \subseteq \mathbb{R}^2$, the unit normal vector of the curve is defined as:

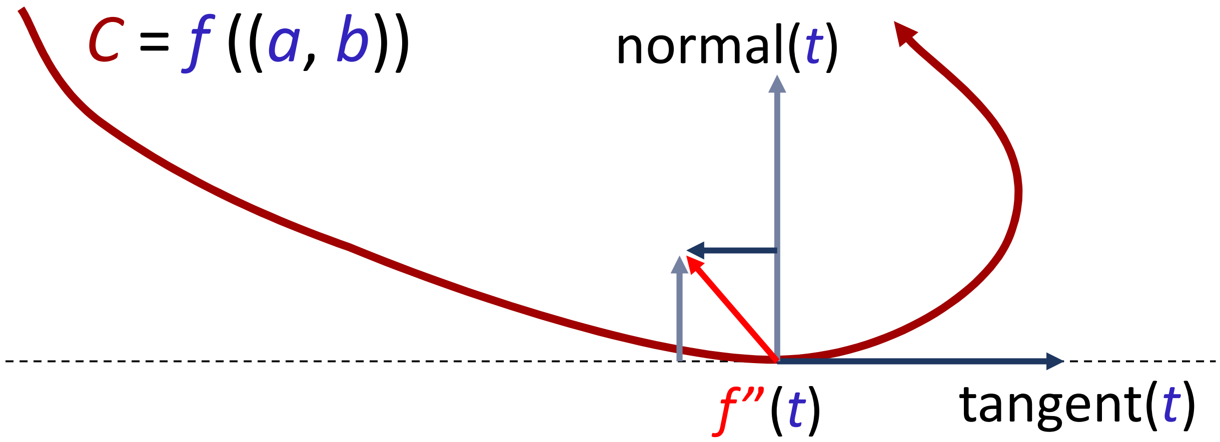
$$\text{normal}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{f'(t)}{\left\| f'(t) \right\|}$$

Curvature

Curvature:

- First derivatives show curve direction / speed of movement.
- Curvature is encoded in 2nd order information.
- Why not just use f''?
- Two problems:
  - Depends on parametrization (different velocity yields different results)
  - Have to distinguish between acceleration in tangential and non-tangential directions.

Curvature & 2nd Derivatives



Definition of curvature

- We want only the non-tangential component of f''.
- Braking / accelerating does not matter for curvature of the traced out curve C.
- Need to normalize speed.

Curvature

Curvature of a Curve C ∈ $\mathbb{R}^2$:

$$\boldsymbol{\kappa 2}(t) = \frac{\left\langle f''(t), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f'(t) \right\rangle}{\|f'(t)\|^3}$$

- Normalization factor:
  - Divide by $\|f'\|$ to obtain unit tangent vector
  - Divide again twice to normalize f''
    - Taylor expansion / chain rule:

$$f(\lambda t) = f(t_0) + \lambda f'(t_0)(t - t_0) + \frac{1}{2}\lambda^2 f''(t)(t - t_0)^2 + O(t^3)$$

    - Second derivative scales quadratically with speed

Unit-speed parametrization

Unit-speed parametrization:

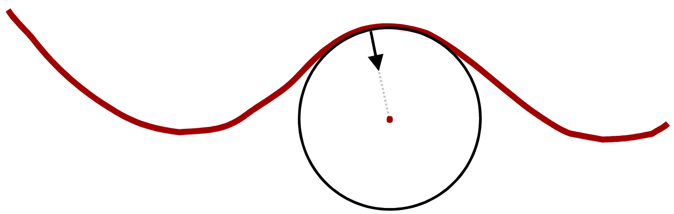
- Assume a unit-speed parametrization, i.e. $\|f'\| \equiv 1$.
- Then, κ2 simplifies to:

$$\boldsymbol{\kappa 2}(t) = \|f''(t)\|$$

Radius of Curvature

Easy to see:

- Curvature of a circle is constant, $\kappa 2 \equiv \pm 1/r$ (r = radius). (see problem sets)
- Accordingly: Define radius of curvature as $1/\kappa 2$.
- Osculating circle:
  - Radius: $1/\kappa 2$
  - Center: $f(t) + \frac{1}{\kappa 2} normal(t)$



Theorems

Definition:

- Rigid motion: $\mathbf{x} \rightarrow \mathbf{A}\mathbf{x}+\mathbf{b}$ with orthogonal $\mathbf{A}$
  - Orientation preserving (no mirroring) if det($\mathbf{A}$) = +1
  - Mirroring leads to det($\mathbf{A}$) = -1

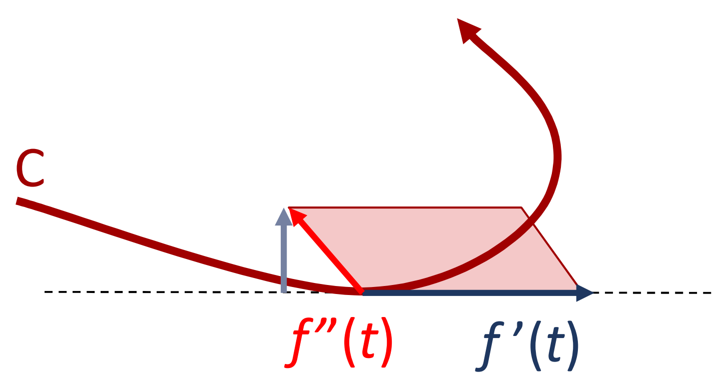
Theorems for plane curves:

- Curvature is invariant under rigid motion
  - Absolute value is invariant
  - Signed value is invariant for orientation preserving rigid motion
- Two unit speed parameterized curves with identical signed curvature function differ only in a orientation preserving rigid motion.

Space Curves

General case: Curvature of a Curve $C \subseteq \mathbb{R}^n$

- *W.l.o.g.*: Assume we are given a unit-speed parametrization f of C
- The *curvature* of C at parameter value t is defined as:
  $$\boldsymbol{\kappa}(t) = \left\| f''(t) \right\|$$
- For a general, regular curve $C \subseteq \mathbb{R}^3$ (any regular parametrization):
  $$\boldsymbol{\kappa}(t) = \frac{\left\| f'(t) \times f''(t) \right\|}{\left\| f'(t) \right\|^3}$$
- General curvature is unsigned



Torsion

Characteristics of Space Curves in $\mathbb{R}^3$:

- Curvature not sufficient
- Curve may “bend” in space
- Curvature is a 2nd order property
- 2nd order curves are always flat
  - Quadratic curves are specified by 3 points in space, which always lie in a plane
  - Cannot capture out-of-plane bends
- Missing property: Torsion

Torsion

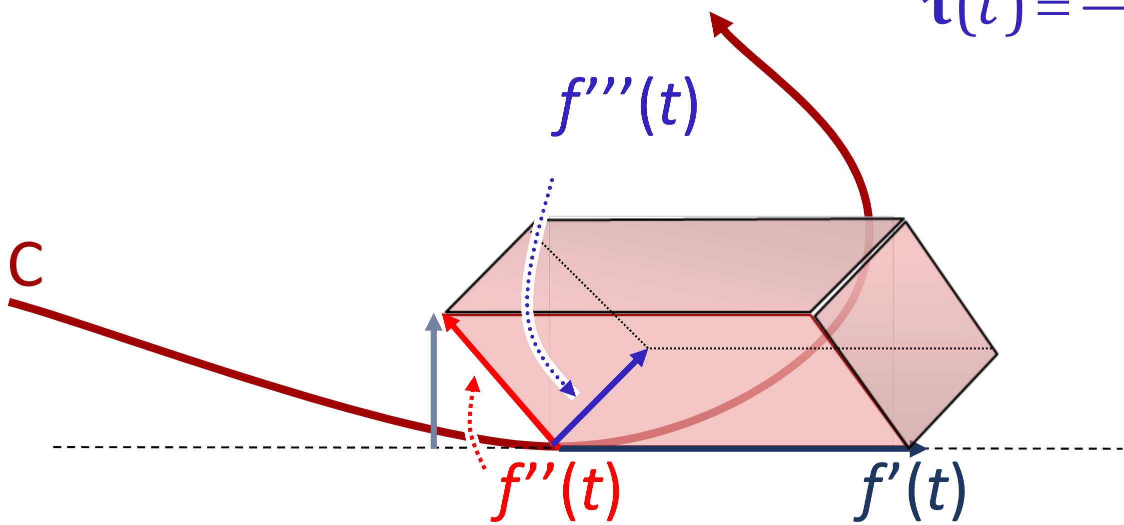
Definition:

- Let f be a regular parametrization of a curve $C \subseteq \mathbb{R}^3$ with non-zero curvature
- The torsion of f at t is defined as

$$\boldsymbol{\tau}(t) = \frac{f'(t) \times f''(t) \cdot f'''(t)}{\left\| f'(t) \times f''(t) \right\|^2} = \frac{\det\left(f'(t), f''(t), f'''(t)\right)}{\left\| f'(t) \times f''(t) \right\|^2}$$

Illustration

$$\boldsymbol{\tau}(t) = \frac{\det\left(f'(t), f''(t), f'''(t)\right)}{\left\| f'(t) \times f''(t) \right\|^2}$$



Theorem

Fundamental Theorem of Space Curves

- Two unit speed parameterized curves $C \subseteq \mathbb{R}^3$ with identical, positive curvature and identical torsion are identical up to a rigid motion.

Part II: Surfaces

Parametric Patches

Parametric Surface Patches:

A smoothly differentiable function

$$f\colon \mathbb{R}^2 \supseteq \Omega \rightarrow \mathbb{R}^n$$

describes a *parametric surface patch*

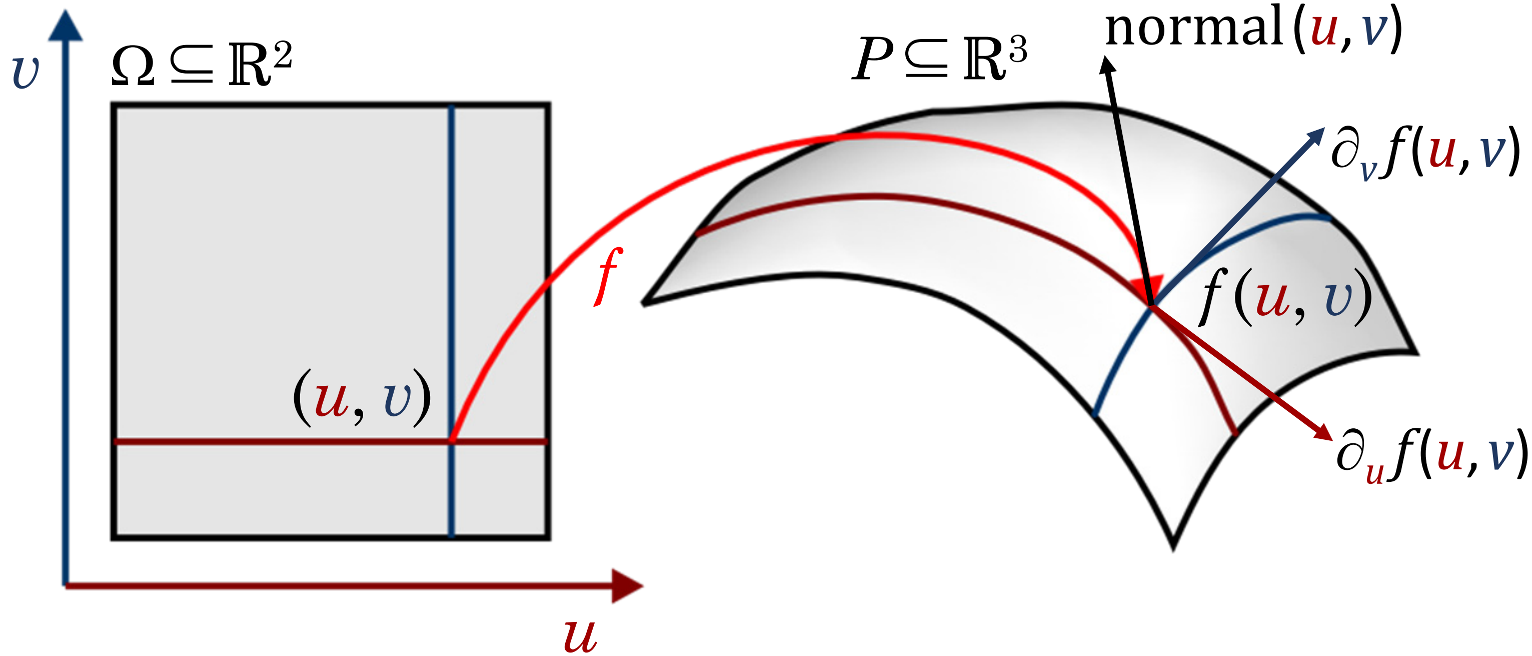
$$P = f(\Omega),\ P \subseteq \mathbb{R}^n.$$

Parametric Patches

Parametric Surface Patches:

- The vectors $\mathrm{tangent}_{\mathbf{x}_0}(r) = \frac{d}{dt} f(\mathbf{x}_0 + t\mathbf{r}) = \nabla_{\mathrm{r}} f(x_0)$ are *tangent vectors* of the surface. In particular, there are canonical tangents $\partial_u f(u,v), \partial_v f(u,v)$ in principal parameter directions.
- *Regular parametrization*: $\partial_u f, \partial_v f$ linearly independent.
- For a regularly parametrized patch in $\mathbb{R}^3$, the unit normal vector is given by: $$\mathrm{normal}(u,v) = \frac{\partial_u f(u,v) \times \partial_v f(u,v)}{\left\| \partial_u f(u,v) \times \partial_v f(u,v) \right\|}$$

Illustration



Tangents

Computing Tangents:

- General tangents can be computed from principal tangents:

$$\text{tangent}_{\mathbf{x}_0}(\mathbf{r}) = \nabla f(\mathbf{x}_0)\mathbf{r} = \begin{pmatrix} | & | \\ \partial_u f(\mathbf{x}_0) & \partial_v f(\mathbf{x}_0) \\ | & | \end{pmatrix} \begin{pmatrix} r_u \\ r_v \end{pmatrix}$$

Surface Area

Surface Area:

- Computation is simple
- For a patch f: $\mathbb{R}^2 \supseteq \Omega \rightarrow \mathbb{R}^n$, integrate over a constant function (one everywhere) over the surface area:
- Then just apply integral transformation theorem:

$$\operatorname{area}(P) = \int_{\Omega} \left\| \partial_u f(\mathbf{x}) \times \partial_u f(\mathbf{x}) \right\| d\mathbf{x}, \quad x = \begin{pmatrix} u \\ v \end{pmatrix}$$

Fundamental Forms

Fundamental Forms:

- Describe the local parametrized surface
- Measure...
  - ...distortion of length (first fundamental form)
  - ...surface curvature (second fundamental form)
- Parametrization independent surface curvature measures will be derived from this

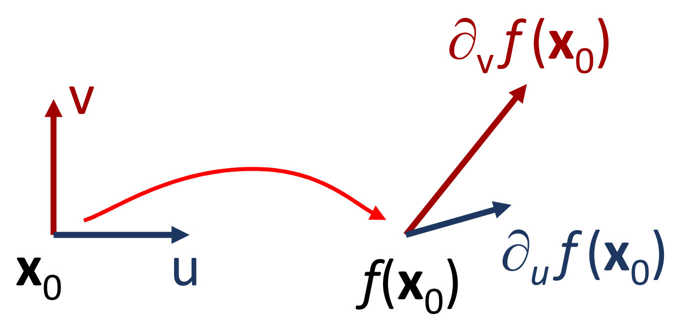
First Fundamental Form

First Fundamental Form

- Also known as *metric tensor*.
- Given a regular parametric patch f: $\mathbb{R}^2 \supseteq \Omega \to \mathbb{R}^3$.
- f will distort angles and distances
- We will look at a local first order Taylor approximation to measure the effect:

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

- Length changes become visible in the scalar product...

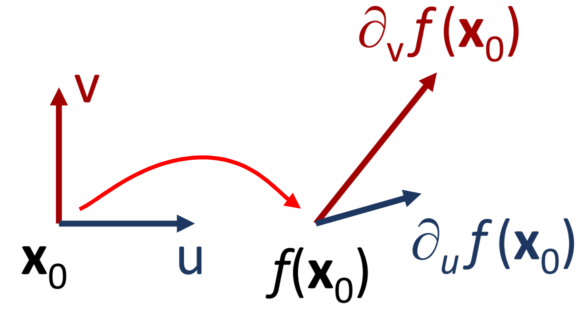


First Fundamental Form

First Fundamental Form

- First order Taylor approximation:

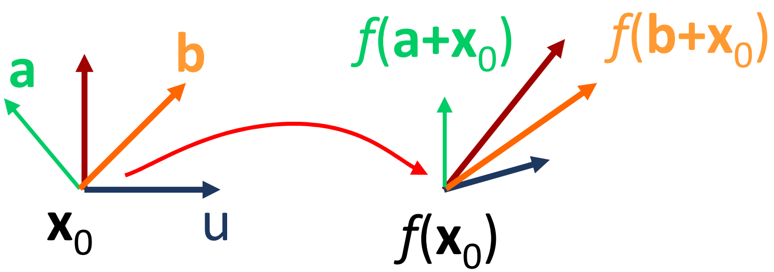
$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$



- Scalar product of vectors a, b $\in \mathbb{R}^2$:

$$\langle f(\mathbf{x}_0 + \mathbf{a}) - f(\mathbf{x}_0), f(\mathbf{x}_0 + \mathbf{b}) - f(\mathbf{x}_0)\rangle \approx \langle \nabla f(\mathbf{x}_0)\mathbf{a}, \nabla f(\mathbf{x}_0)\mathbf{b}\rangle$$

$$= \mathbf{a}^{\mathrm{T}} \underbrace{\left(\nabla f(\mathbf{x}_0)^{\mathrm{T}} \nabla f(\mathbf{x}_0)\right)}_{\text{first fundamental form}} \mathbf{b}$$



First Fundamental Form

First Fundamental Form

- The first fundamental form can be written as a 2×2 matrix:

$$\left(\nabla f^{\mathrm{T}} \nabla f\right) = \begin{pmatrix} \partial_{\mathbf{u}} f \, \partial_{\mathbf{u}} f & \partial_{\mathbf{u}} f \, \partial_{\mathbf{v}} f \\ \partial_{\mathbf{u}} f \, \partial_{\mathbf{v}} f & \partial_{\mathbf{v}} f \, \partial_{\mathbf{v}} f \end{pmatrix} =: \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \mathbf{I}(\mathbf{x}, \mathbf{y}) := \mathbf{x}^{\mathrm{T}} \left(\nabla f^{\mathrm{T}} \nabla f\right) \mathbf{y}$$

- The matrix is symmetric and positive definite (for a regular parametrization)
- Defines a *generalized scalar product* that measures lengths and angles *on the surface*.

Second Fundamental Form

Problems:

- The first fundamental form measures length changes only
- A cylinder looks like a flat sheet in this view
- We need a tool to measure curvature of a surface as well
- Again, we will need second order information
  (any first order approximation is inherently flat)

Second Fundamental Form

Definition:

- Given a regular parametric patch *f*: $\mathbb{R}^2 \supseteq \Omega \rightarrow \mathbb{R}^3$.
- The *second fundamental* form (also known as *shape operator*, or *curvature tensor*) is the matrix:

$$S(\mathbf{x}_0) = \begin{pmatrix} \partial_{\mathbf{uu}} f(\mathbf{x}_0) \cdot \mathbf{n} & \partial_{\mathbf{uv}} f(\mathbf{x}_0) \cdot \mathbf{n} \\ \partial_{\mathbf{uv}} f(\mathbf{x}_0) \cdot \mathbf{n} & \partial_{\mathbf{vv}} f(\mathbf{x}_0) \cdot \mathbf{n} \end{pmatrix}$$

- Notation:

$$\mathbf{II}(\mathbf{x},\mathbf{y}) = \mathbf{x}^{\mathrm{T}} \begin{pmatrix} \partial_{\mathbf{uu}} f(\mathbf{x}_0) \cdot \mathbf{n} & \partial_{\mathbf{uv}} f(\mathbf{x}_0) \cdot \mathbf{n} \\ \partial_{\mathbf{uv}} f(\mathbf{x}_0) \cdot \mathbf{n} & \partial_{\mathbf{vv}} f(\mathbf{x}_0) \cdot \mathbf{n} \end{pmatrix} \mathbf{y}$$

Second Fundamental Form

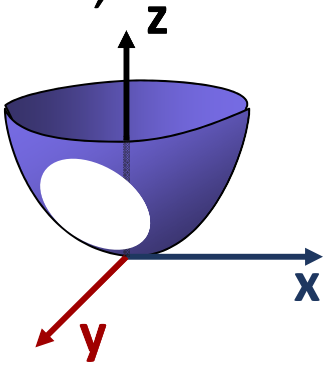
Basic Idea:

- Compute second derivative vectors
- Project in normal direction (remove tangential acceleration)

Alternative Computation

Alternative Formulation (Gauss):

- Local height field parameterization f(**x**,**y**) = **z**
- Orthonormal **x**,**y** coordinates *tangential* to surface, **z** in normal direction, origin at zero
- 2nd order Taylor representation:



$$f(\mathbf{x}) \approx \frac{1}{2} \underbrace{\mathbf{x}^{\mathrm{T}} f''(\mathbf{x})\mathbf{x}} + \underbrace{f'(\mathbf{x})\mathbf{x} + f(0)}_{0}$$
$$= ex^2 + 2fxy + gy^2$$

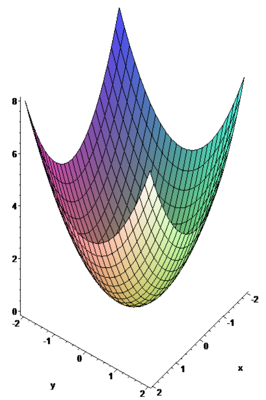
- Second fundamental form: Matrix of second derivatives

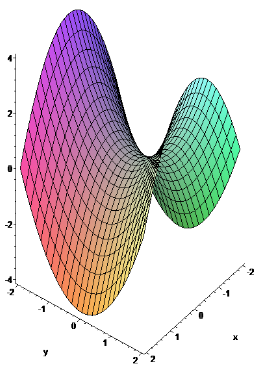
$$\begin{pmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{xy} f & \partial_{yy} f \end{pmatrix} =: \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

Basic Idea

In other words:

- The first fundamental form is the linear part (squared) of local Taylor approximation.
- The second fundamental form is the quadratic part of a local quadratic approximation of the heightfield
- The matrix is symmetric. So next thing to try is eigenanalysis, of course...

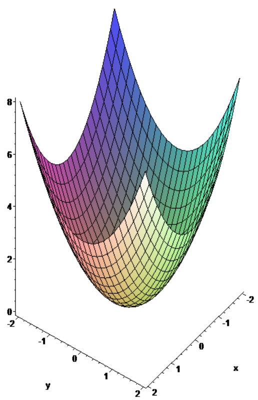




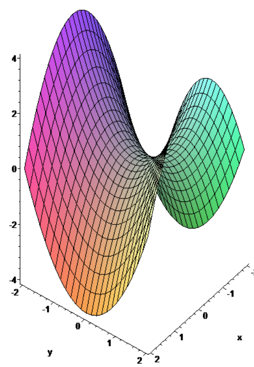
Principal Curvature

Eigenanalysis:

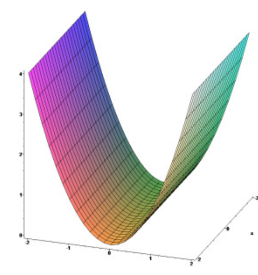
- The eigenvalues of the shape operator for an orthonormal tangent basis are called *principal curvatures* κ_1, κ_2.
- The corresponding eigenvectors (which are orthogonal) are called *principal directions of curvature*.
- Again, we get different cases...:



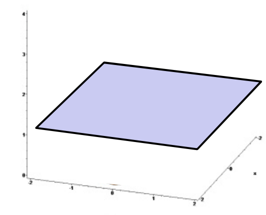
$\kappa_i > 0$



$\kappa_0 > 0, \kappa_1 < 0$



$\kappa_0 = 0, \kappa_1 > 0$



$\kappa_0 = 0, \kappa_1 = 0$

...

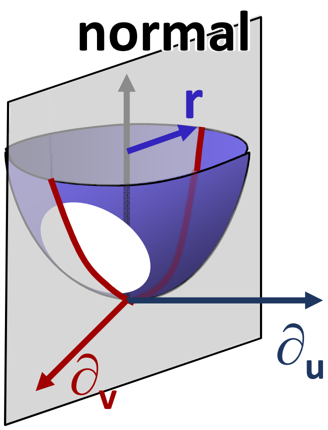
Normal Curvature

Definition:

- The *normal curvature* $\mathbf{k}(\mathbf{r})$ in direction $\mathbf{r}$ for a unit length direction vector $\mathbf{r}$ at parameter position $\mathbf{x}_0$ is given by:

$$\mathbf{k}_{\mathbf{x}_0}(\mathbf{r}) = \mathbf{II}_{\mathbf{x}_0}(\mathbf{r},\mathbf{r}) = \mathbf{r}^{\mathrm{T}}\mathbf{S}(\mathbf{x}_0)\mathbf{r}$$

Relation to Curvature of Plane Curves:



- Intersect the surface locally with plane spanned by **normal** and $\mathbf{r}$ through point $\mathbf{x}_0$.
- The curvature of the curve at $\mathbf{x}_0$ is equal to the normal curvature up to its sign.

Principal Curvatures

Relation to principal curvature:

- The maximum principal cuvature κ_1 is the maximum of the normal curvature
- The minimum principal cuvature κ_2 is the minimum of the normal curvature

Gaussian & Mean Curvature

More Definitions:

- The Gaussian curvature K is the product of the principal curvatures: $K = \kappa_1 \kappa_2$
- The mean curvature H is the average: $H = 0.5 \cdot (\kappa_1 + \kappa_2)$

Theorems:

- $K(\mathbf{x}_0) = \det\big(S(x_0)\big) = \dfrac{eg - f^2}{EG - F^2}$
- $H(\mathbf{x}_0) = \dfrac{1}{2}\operatorname{tr}\big(S(x_0)\big) = \dfrac{eG - 2fF + gE}{2\left(EG - F^2\right)}$

Global Properties

Definition:

- An *isometry* is a mapping between surfaces that preserves distances on the surface (geodesics)
- A *developable surface* is a surface with Gaussian curvature zero everywhere (i.e. no curvature in at least one direction)
  - Examples: Cylinder, Cone, Plane
- A developable surface can be locally mapped to a plane isometrically (flattening out, unroll).

Theorema Egregium

Theorema egregium (Gauss):

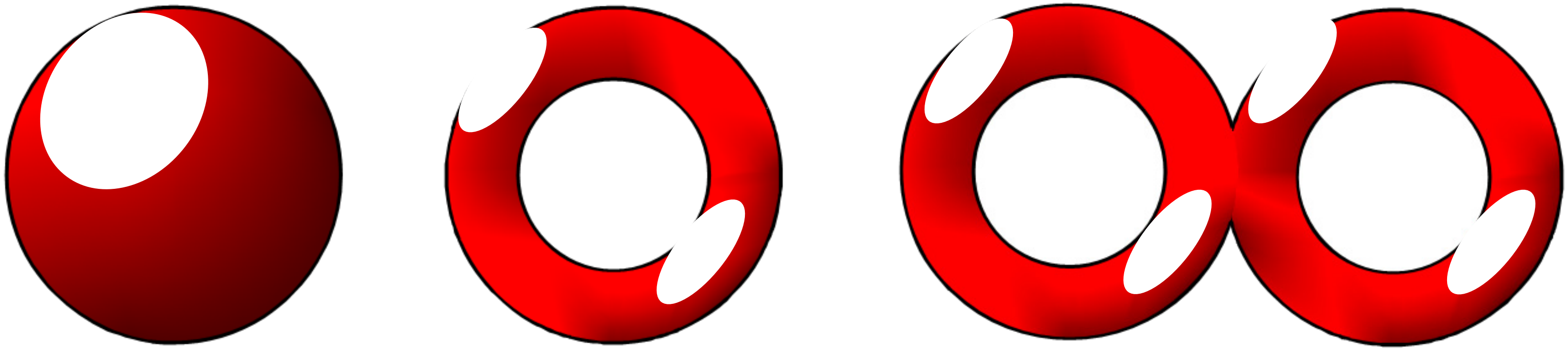
- Any isometric mapping preservers Gaussian curvature, i.e. Gaussian curvature is invariant under isometric maps ("intrinsic surface property")
- Consequence: The earth ($\approx$ sphere) cannot be mapped to a plane in an exactly length preserving way.

Gauss Bonnet Theorem

Gauss Bonnet Theorem:

For a compact, orientable surface without boundary in $\mathbb{R}^3$, the area integral of the Gauss curvature is related to the genus g of the surface:

$$\int_S K(x)dx = 4\pi(1-g)$$



...

g = 0 g = 1 g = 2

Fundamental Theorem of Surfaces

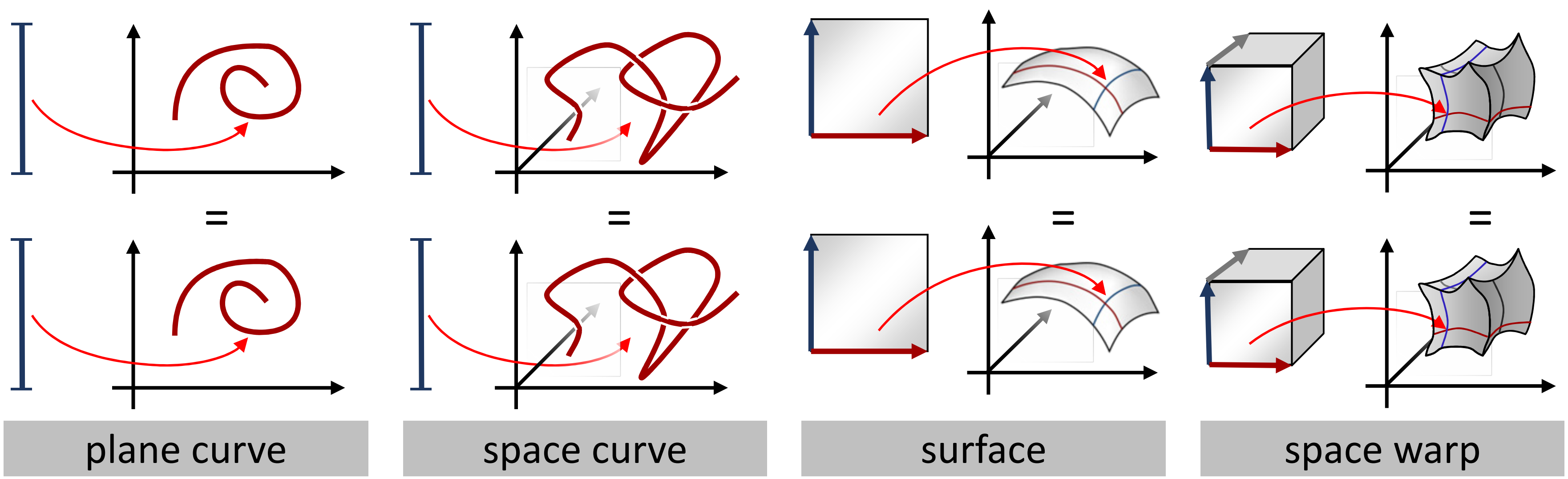
Theorem:

- Given two parametric patches in $\mathbb{R}^3$ defined on the same domain Ω.
- Assume that the first and second fundamental form are identical.
- Then there exists a rigid motion that maps on surface to the other.

Summary

Objects are the same up to a rigid motion, if...:

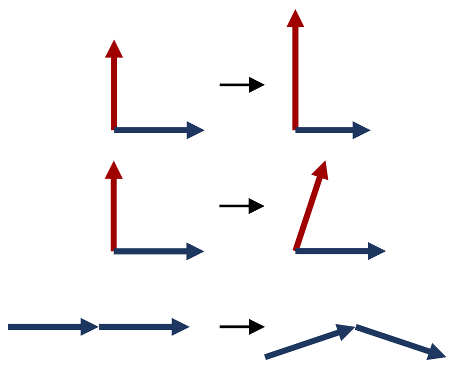
- Curves $\mathbb{R} \to \mathbb{R}^2$: Same *speed*, same *curvature*
- Curves $\mathbb{R} \to \mathbb{R}^3$: Same *speed*, same *curvature*, *torsion*
- Surfaces $\mathbb{R}^2 \to \mathbb{R}^3$: Same *first* & *second* fundamental form
- Volumetric Objects $\mathbb{R}^3 \to \mathbb{R}^3$: Same *first* fundamental form



Deformation Models

What if this does not hold?

- Deviation in fundamental forms is a measure of deformation
- Example: Surfaces
  - Diagonals of $\mathbf{I}_1$ - $\mathbf{I}_2$: scaling (stretching)
  - Off-diagonals of $\mathbf{I}_1$ - $\mathbf{I}_2$: sheering
  - Elements of $\mathbf{II}_1$ - $\mathbf{II}_2$: bending
- This is the basis of *deformation models*.



Reference: D. Terzopoulos, J. Platt, A. Barr, K. Fleischer: Elastically Deformable Models. In: *Siggraph '87 Conference Proceedings (Computer Graphics 21(4))*, 1987.