

Music Processing

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Fourier Transform

Exercise 1

Let $f : [0, 1] \rightarrow \mathbb{R}$, $f \in L^2([0, 1])$. Then, f can be represented in form of the following two variants of the Fourier series:

$$f(t) = d_0 + \sum_{k \in \mathbb{N}} d_k \sqrt{2} \cos(2\pi(kt - \phi_k)) \quad (1)$$

$$f(t) = a_0 + \sum_{k \in \mathbb{N}} a_k \sqrt{2} \cos(2\pi kt) + \sum_{k \in \mathbb{N}} b_k \sqrt{2} \sin(2\pi kt) \quad (2)$$

Show that $d_0 = a_0$, $d_k = \sqrt{a_k^2 + b_k^2}$, and $\phi_k = \frac{1}{2\pi} \arccos\left(\frac{a_k}{d_k}\right)$ for $k \in \mathbb{N}$.

Solution

We use the addition theorem $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ with $\alpha = 2\pi kt$, $\beta = -2\pi \phi_k$. Then

$$\begin{aligned} \cos(2\pi(kt - \phi_k)) &= \cos(2\pi kt) \cos(-2\pi \phi_k) - \sin(2\pi kt) \sin(-2\pi \phi_k) \\ &= \cos(2\pi \phi_k) \cos(2\pi kt) + \sin(2\pi \phi_k) \sin(2\pi kt) \end{aligned}$$

Comparing coefficients in (1) and (2) one obtains

$$\left. \begin{aligned} a_k &= d_k \cos(2\pi \phi_k) \\ b_k &= d_k \sin(2\pi \phi_k) \end{aligned} \right\} \Rightarrow (a_k^2 + b_k^2) = d_k^2,$$

and $\cos(2\pi \phi_k) = \frac{a_k}{d_k}$ implies $\phi_k = \frac{1}{2\pi} \arccos\left(\frac{a_k}{d_k}\right)$.

Exercise 2

Let $f : [0, 1] \rightarrow \mathbb{R}$, $f \in L^2([0, 1])$, be as in the previous exercise. Then, using a complex formulation of the Fourier series, f can be represented as

$$f(t) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t}. \quad (3)$$

Let $c_k = |c_k| e^{2\pi i \gamma_k}$ be the polar coordinate representation of the complex number c_k . Show that

$$\begin{aligned} d_0 &= c_0, & d_k &= \sqrt{2}|c_k|, & \phi_k &= \gamma_k, \\ a_k &= \sqrt{2} \operatorname{Re}(c_k), & b_k &= -\sqrt{2} \operatorname{Im}(c_k). \end{aligned}$$

for $k \in \mathbb{N}$.

Solution

Since f is a real-valued function, one obtains

$$\sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t} = f(t) = \overline{f(t)} = \overline{\sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t}} = \sum_{k \in \mathbb{Z}} \overline{c_k} e^{-2\pi i k t}$$

and therefore $c_{-k} = \overline{c_k}$. From this follows

$$\begin{aligned} f(t) &= \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t} \\ &= c_0 + \sum_{k \in \mathbb{N}} (c_k e^{2\pi i k t} + c_{-k} e^{-2\pi i k t}) \\ &= c_0 + \sum_{k \in \mathbb{N}} (c_k e^{2\pi i k t} + \overline{c_k} \overline{e^{2\pi i k t}}) \\ &= c_0 + \sum_{k \in \mathbb{N}} 2 \operatorname{Re}(c_k e^{2\pi i k t}) \\ &= c_0 + \sum_{k \in \mathbb{N}} 2 \operatorname{Re} \left((\operatorname{Re}(c_k) + i \operatorname{Im}(c_k)) \cdot (\cos(2\pi k t) + i \sin(2\pi k t)) \right) \\ &= c_0 + \sum_{k \in \mathbb{N}} 2 \operatorname{Re}(c_k) \cos(2\pi k t) - 2 \operatorname{Im}(c_k) \sin(2\pi k t). \end{aligned} \quad (4)$$

Hence, by comparing coefficients of (4) with (2), one obtains

$$\begin{aligned} a_k &= \sqrt{2} \operatorname{Re}(c_k), & b_k &= -\sqrt{2} \operatorname{Im}(c_k) \\ d_k &= \sqrt{a_k^2 + b_k^2} = \sqrt{2}|c_k|. \\ d_0 &= c_0. \end{aligned}$$

Moreover, by substituting a_k and d_k , one obtains

$$\begin{aligned} \phi_k &= \frac{1}{2\pi} \arccos \left(\frac{a_k}{d_k} \right) \\ &= \frac{1}{2\pi} \arccos \left(\frac{\operatorname{Re}(c_k)}{|c_k|} \right) \\ &= \frac{1}{2\pi} \arccos \left(\frac{|c_k| \cos(2\pi \gamma_k)}{|c_k|} \right) = \gamma_k. \end{aligned}$$