Geometric Modeling

Summer Semester 2012

Triangle Meshes and Multi-Resolution Representations

Representations · Hierarchical Data Structures · Rendering

UNIVERSITÄT DES SAARLANDES

CLUSTER OF EXCELLENCE

max planck institut informatik

Overview...

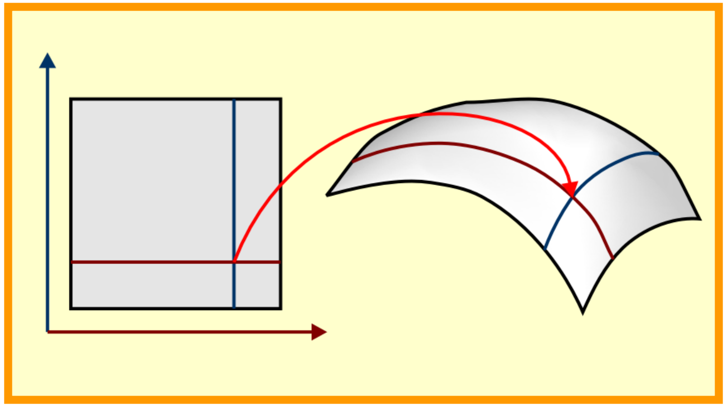
Topics:

- Blossoming and Polars
- Rational Spline Curves
- Spline Surfaces
- Triangle Meshes & Multi-Resolution Representations
  - Mesh Data Structures
  - Triangulations
  - Spatial Data Structures and Algorithms
  - Mesh Simplification
  - Appearance Approximation

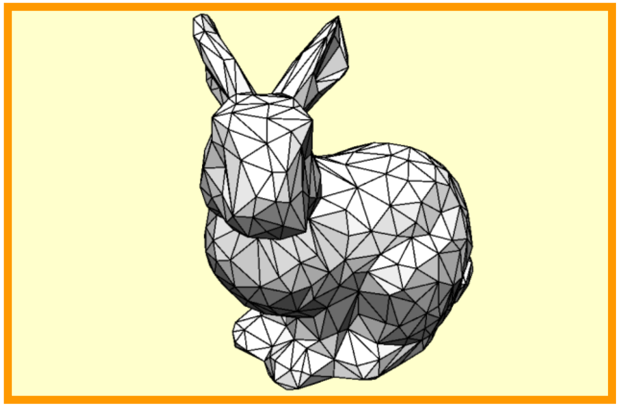
Triangle Meshes

Data Structures

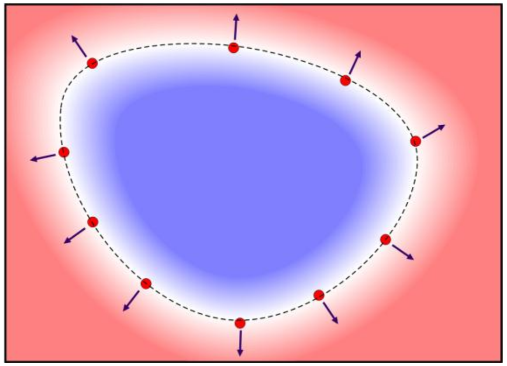
Modeling Zoo



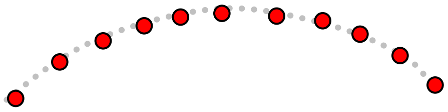
Parametric Models



Primitive Meshes



Implicit Models



Particle Models

Triangle Meshes

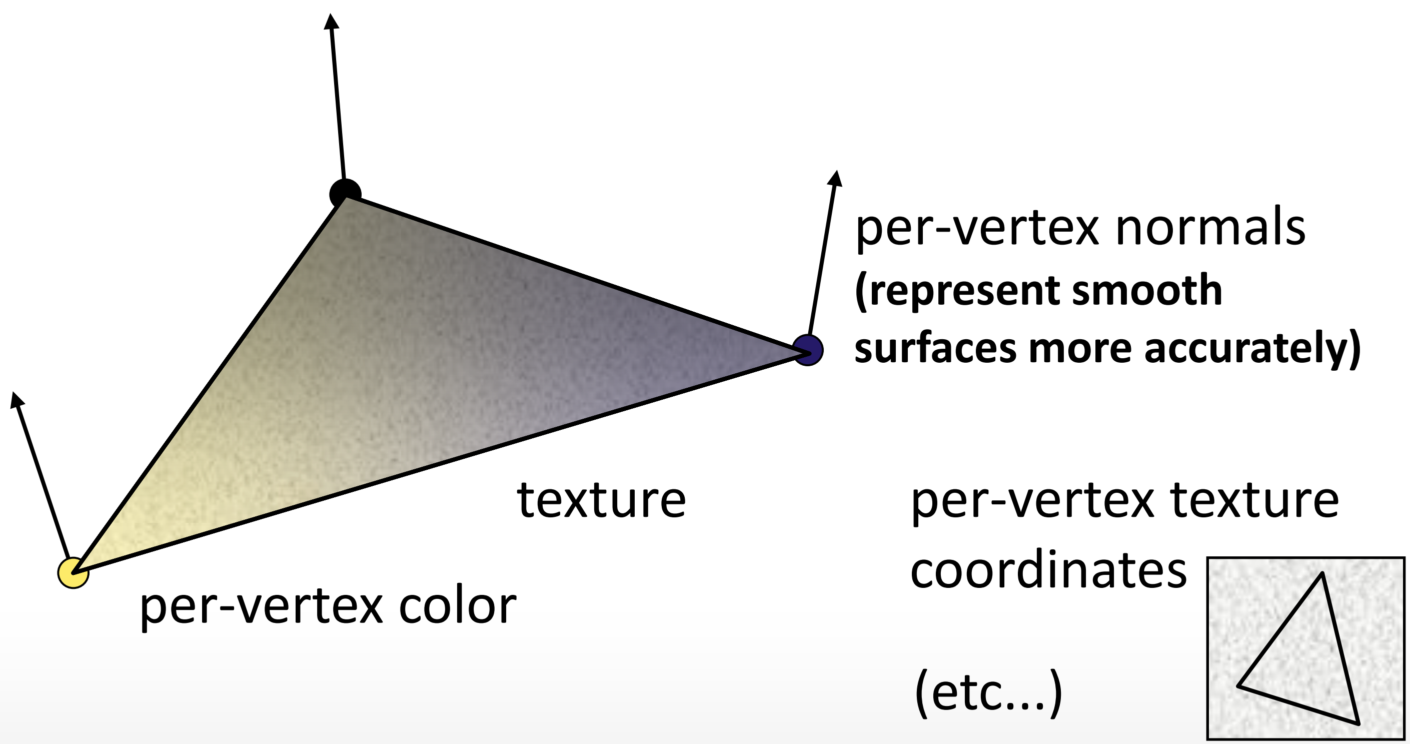
Triangle Meshes:

- Triangle meshes are probably the most common surface representation in computer graphics
- Triangles are probably the simplest surface primitives that can be assembled into meshes
  - Rendering can be implemented in hardware (z-buffering)
  - Simple algorithms for intersections (raytracing, collisions)

Attributes

How to define a triangle?

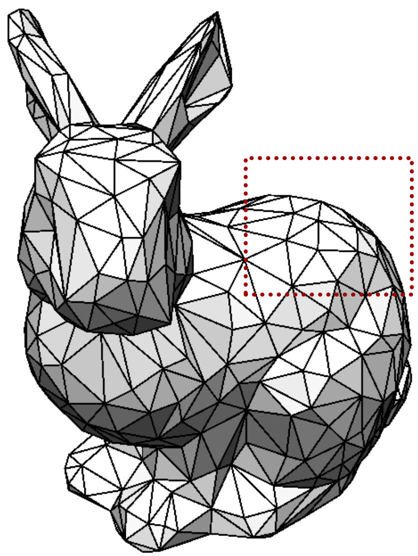
- We need three points in $\mathbb{R}^3$ (obviously).
- But we can have more:

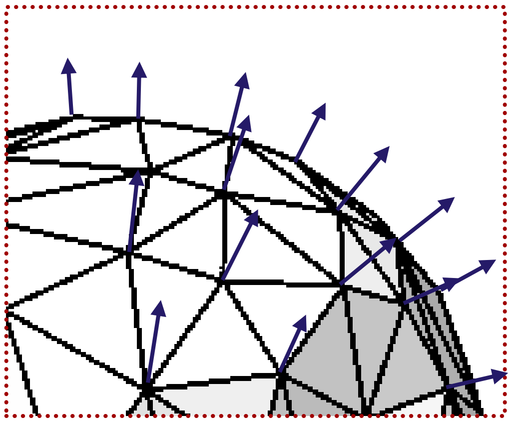


Shared Attributes in Meshes

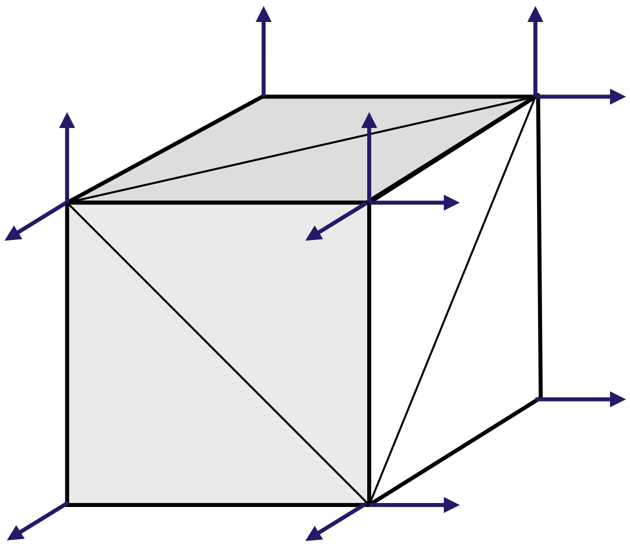
In Triangle Meshes:

- Attributes might be shared or separated:





adjacent triangles
share normals



adjacent triangles
have separated normals

"Triangle Soup"

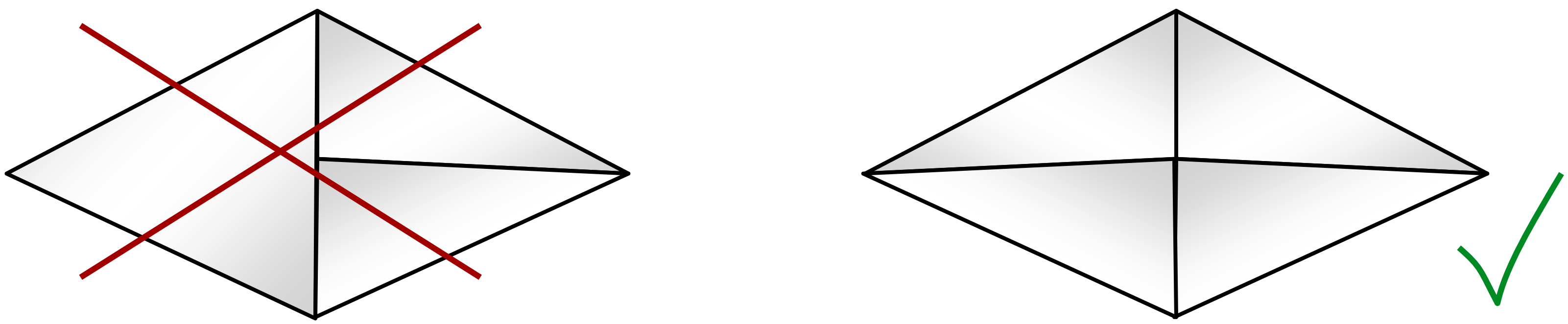
Variants in triangle mesh representations:

- "*Triangle Soup*"
  - A set $S = \{t_1, ..., t_n\}$ of triangles
  - No further conditions
  - This is "the most common" representation (if you download models from the web, you never know what you get)
- *Triangle Meshes*: Additional consistency conditions
  - Conforming meshes: Vertices meet only at vertices
  - Manifold meshes: No intersections, no T-junctions

Conforming Meshes

Conforming Triangulation:

- Vertices of triangles must only meet at vertices, not in the middle of edges:

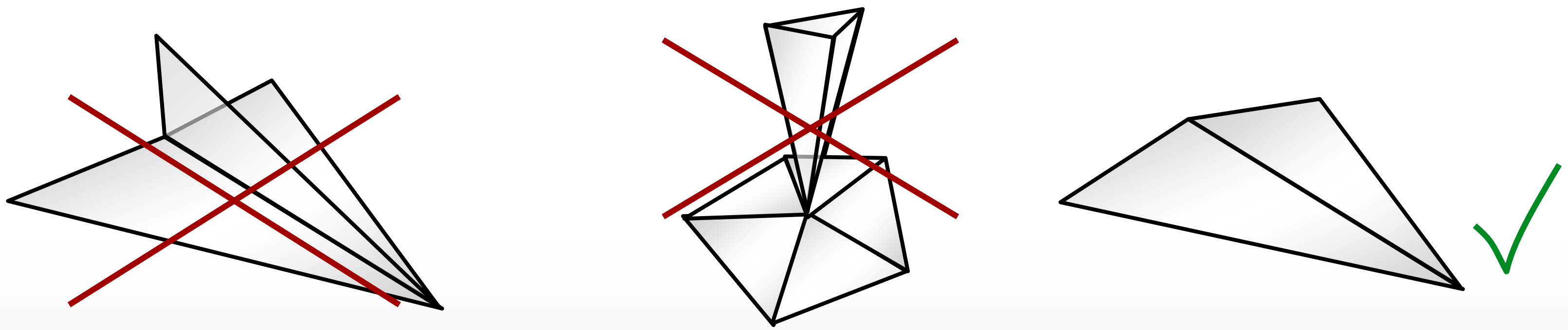


- This makes sure that we can move vertices around arbitrarily without creating holes in the surface

Manifold Meshes

Triangulated two-manifold:

- Every edge is incident to exactly 2 triangles (closed manifold)
- ...or to at most two triangles (manifold with boundary)
- No triangles intersect (other than along common edges or vertices)
- Two triangles that share a vertex must share an edge



Attributes

In general:

- Vertex attributes:
  - Position (mandatory)
  - Normals
  - Color
  - Texture Coordinates
- Face attributes:
  - Color
  - Texture
- Edge attributes (rarely used)
  - E.g.: Visible line

Data Structures

The simple approach: List of vertices, edges, triangles

$\mathtt{v}_1$: (posx posy posy), $\mathtt{attrib}_1$, ..., $\mathtt{attrib}_{nav}$

...

$\mathtt{v}_{nv}$: (posx posy posy), $\mathtt{attrib}_1$, ..., $\mathtt{attrib}_{nav}$

$\mathtt{e}_1$: ($\mathtt{index}_1$ $\mathtt{index}_2$), $\mathtt{attrib}_1$, ..., $\mathtt{attrib}_{nae}$

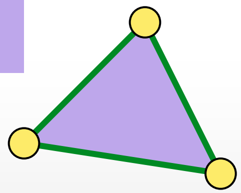
...

$\mathtt{e}_{ne}$: ($\mathtt{index}_1$ $\mathtt{index}_2$), $\mathtt{attrib}_1$, ..., $\mathtt{attrib}_{nae}$

$\mathtt{t}_1$: ($\mathtt{idx}_1$ $\mathtt{idx}_2$ $\mathtt{idx}_3$), $\mathtt{attrib}_1$, ..., $\mathtt{attrib}_{nat}$

...

$\mathtt{t}_{nt}$: ($\mathtt{idx}_1$ $\mathtt{idx}_2$ $\mathtt{idx}_3$), $\mathtt{attrib}_1$, ..., $\mathtt{attrib}_{nat}$



Pros & Cons

Advantages:

- Simple to understand and build
- Provides exactly the information necessary for rendering

Disadvantages:

- Dynamic operations are expensive:
  - Removing or inserting a vertex
    $\rightarrow$ renumber expected edges, triangles
- Adjacency information is one-way
  - Vertices adjacent to triangles, edges $\rightarrow$ direct access
  - Any other relationship $\rightarrow$ need to search
  - Can be improved using hash tables (but still not dynamic)

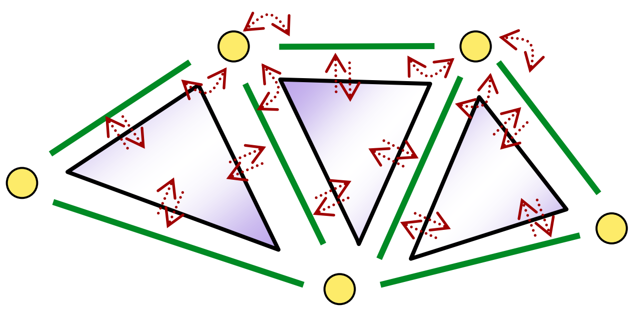
Adjacency Data Structures

Alternative:

- Some algorithms require extensive neighborhood operations (get adjacent triangles, edges, vertices)
- ...as well as dynamic operations (inserting, deleting triangles, edges, vertices)
- For such algorithms, an *adjacency based* data structure is usually more efficient
  - The data structure encodes the graph of mesh elements
  - Using pointers to neighboring elements

First try...

Straightforward Implementation:

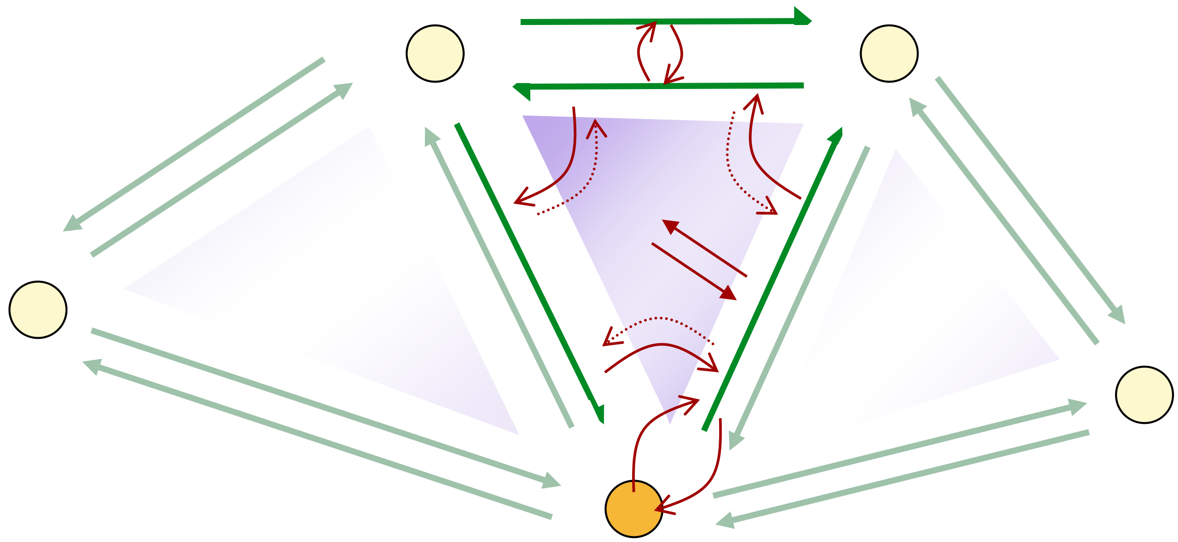


- Use a list of vertices, edges, triangles
- Add a pointer from each element to each of its neighbors
- Global triangle list can be used for rendering

Remaining Problems:

- Lots of redundant information – hard to keep consistent
- Adjacency lists might become very long
  - Need to search again (might become expensive)
  - This is mostly a "theoretical problem" (O(n) search)

Less Redundant Data Structures



Half edge data structure:

- Half edges, connected by clockwise / ccw pointers
- Pointers to opposite half edge
- Pointers to/from start vertex of each edge
- Pointers to/from left face of each edge

Implementation

```
// a half edge
struct HalfEdge {
    HalfEdge* next;
    HalfEdge* previous;
    HalfEdge* opposite;

    Vertex* origin;
    Face* leftFace;
    EdgeData* edge;
};

// the data of the edge
// stored only once
struct EdgeData {
    HalfEdge* anEdge;
    /* attributes */
};
```

```
// a vertex
struct Vertex {
    HalfEdge* someEdge;
    /* vertex attributes */
};

// the face (triangle, poly)
struct Face {
    HalfEdge* half;
    /* face attributes */
};
```

Implementation

Implementation:

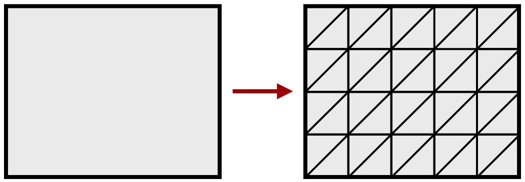
- The half-edge data structure
  - Less redundant representation of the mesh
  - Relatively easy to implement
  - A lot of mesh operations can be performed faster
- Free Implementations are available, for example
  - OpenMesh
  - CGAL
- Alternative data structures: for example winged edge (Baumgart 1975)

Triangulations

Algorithms and Data Structures

Triangulation

Problem Statement:



- Given a 2-dimensional domain
- We want to triangulate the domain
- We need this for example for rendering parametric surfaces by triangle rasterization
- Adaptive triangulation: Higher resolution in more important area

Different Problem:

- Triangulating a point cloud in $\mathbb{R}^3$
- This is the surface reconstruction problem (we will look at that later)

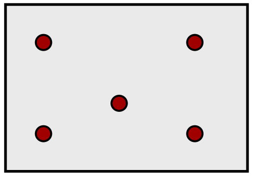
Problem Variations

Simplest Version

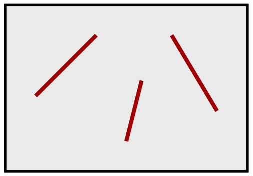
- Domain is a rectangle or a triangle
- Uniform or adaptive tessellation

More Complex: Constrained Triangulation

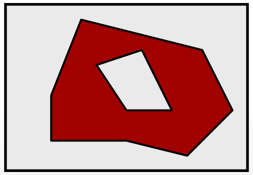
- Point constraints:
  specific points must be included



- Edge constraints:
  specific edges must be included



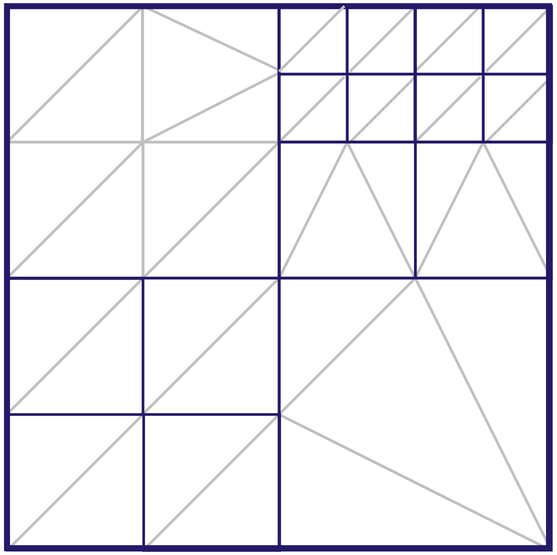
- Boundary constraints:
  triangulate within some area only

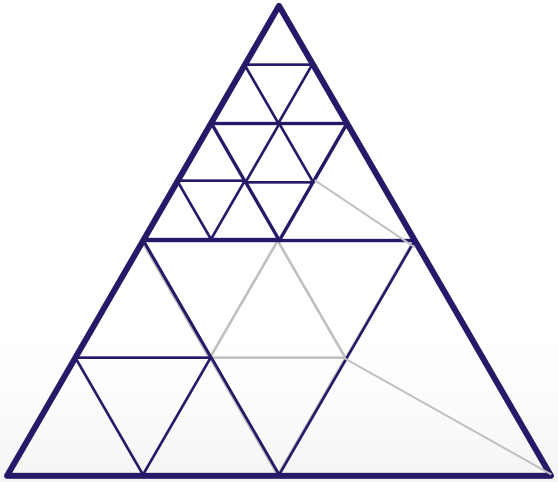


Adaptive Triangulation

Unconstrained adaptive triangulation:

- Hierarchy of rectangles / triangles (Quadtree), 1-to-4 split
- Use “balancing” to limit depth differences
- Balancing will increase the number of nodes in the tree by a factor of at most O(1)
- Finally, create a conforming triangulation (fixed number of cases per node)

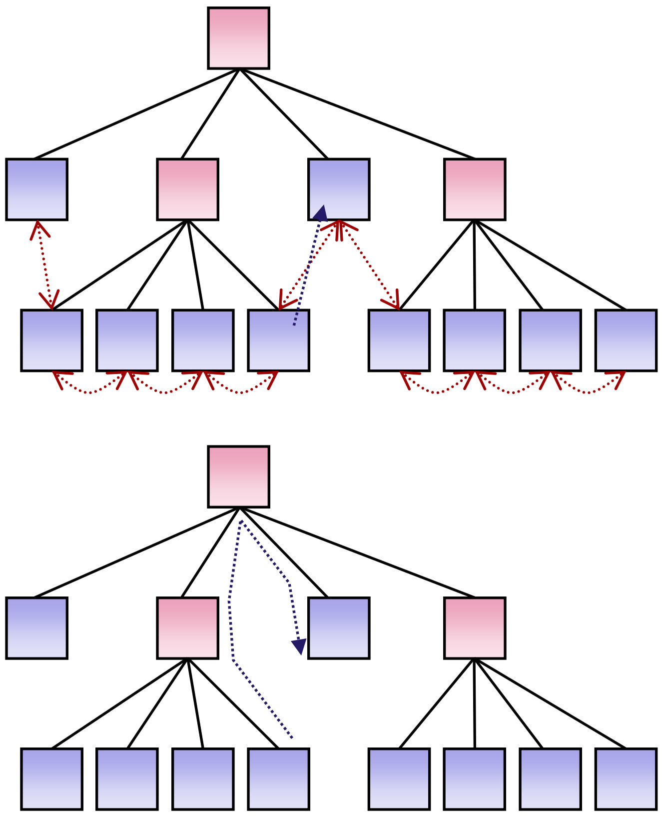




Implementation

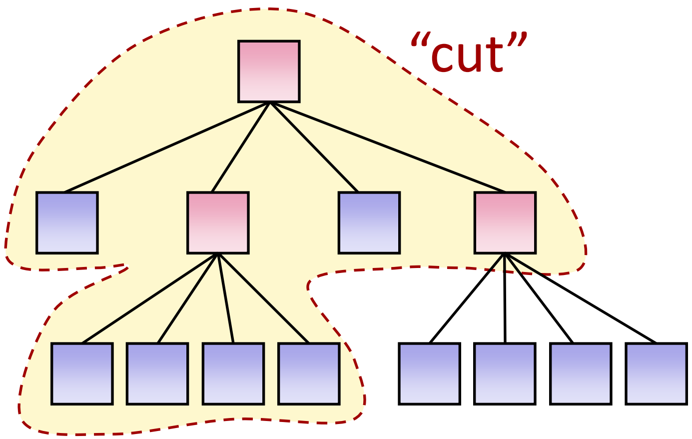
Storage: Tree Structure

- Tree can be represented directly
- Neighbor search for balancing:
  - We can store fixed pointers to neighboring cells
    (not that elegant, easy to mess up the consistency)
  - Alternative: use neighborhood search
    - Go up in tree until common ancestor is found
    - Then go down again
    - O(1) expected running time



Adaptive Rendering

Adaptive rendering algorithm

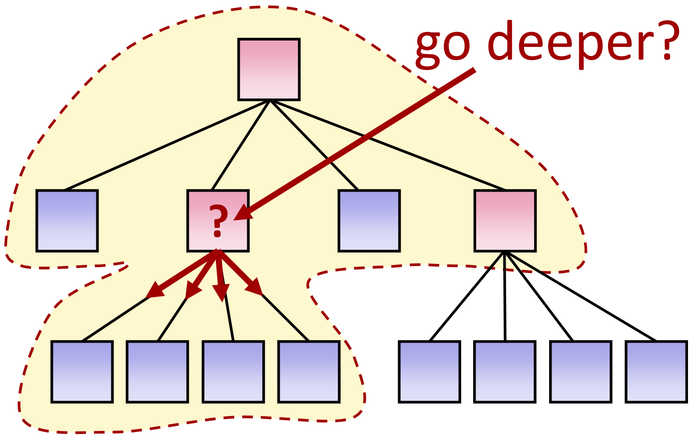


- Recursive algorithm
- Starts at root node
- Is precision sufficient?
  - If so → stop recursion
  - Otherwise → go to child nodes
- The recursion extracts a subgraph of the tree (“cut”)
- Next: The subgraph needs to be balanced
- Then, a triangulation can be created

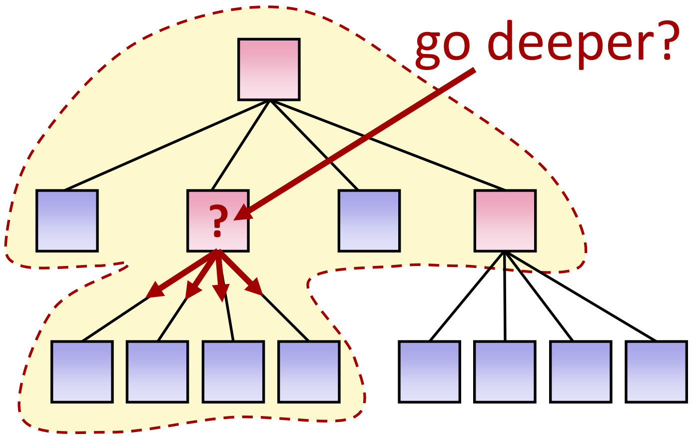
Adaptive Rendering

Termination Criteria:

- Rendering error:
  - Projected size on screen shrinks with $1/z$ (where z is the depth in camera coordinates)
  - Might also depend on viewing angle (typically, this is neglected)
- Geometric error:
  - Tessellating a curved surface with planar faces is only an approximation
  - Error depends on curvature



Adaptive Rendering



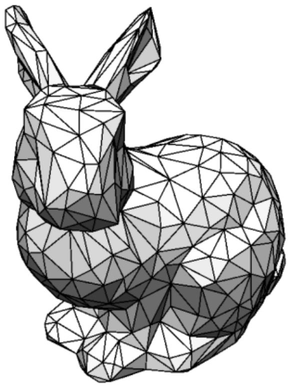
Termination Criteria:

- Typically: divide geometric error by *z*
- To estimate *z*, use a bounding box (for splines: convex hull property)
- Chooses nearest *z* (conservative estimate)
- REYES algorithm [Cook, Carpenter, Catmull 1987] (Pixar's RenderMan)
  - Stop subdivision when BB below one pixel on screen size
  - Subdivision connectivity not really necessary in that case

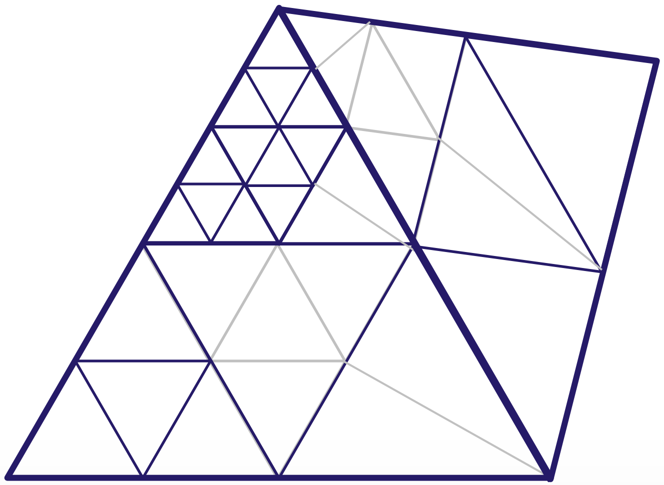
Subdivision Connectivity Meshes

Generalization: Arbitrary Domains

- Start with a base mesh
  - "3D parametrization"
  - A conforming two-manifold mesh in 3D used as parametrization domain
- The base mesh fixes the topology
- Subdivide recursively as needed
- Now: Balancing/triangulation, *also across borders*
- Then compute the final surface



base mesh



consistency across boundaries

Hardware Friendly Version

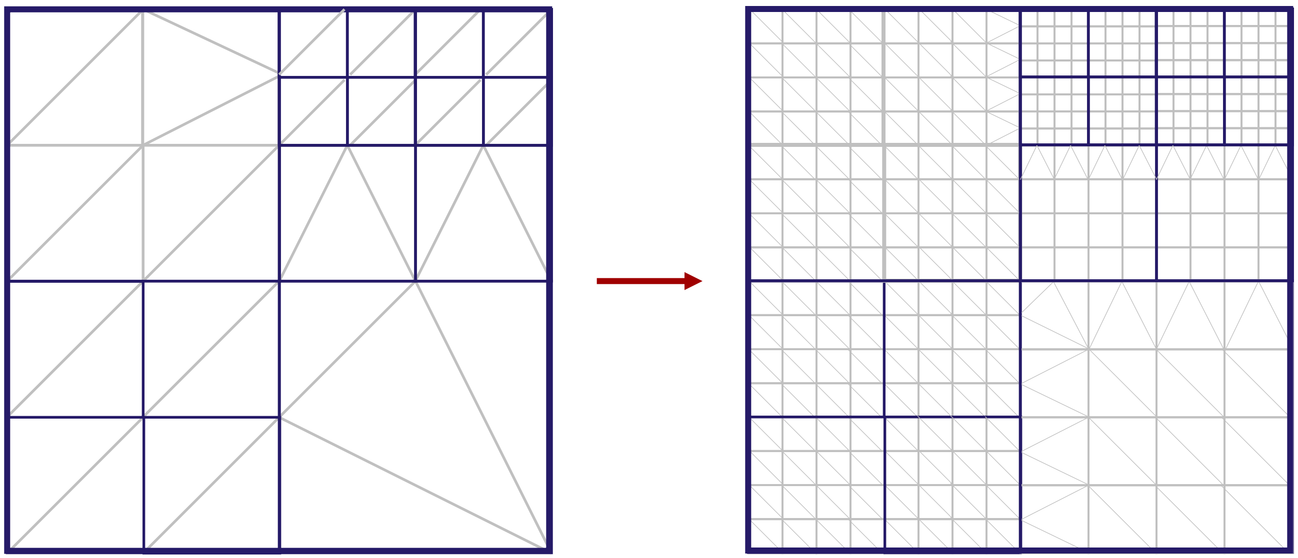
Problems:

- Costs for hierarchy creation / balancing are quite large
- In particular: Problematic for rendering
- Rendering triangles is very cheap these days
- But we still need adaptivity (moving camera, we can get arbitrarily close)
- Solution: *Subdivision connectivity grids*

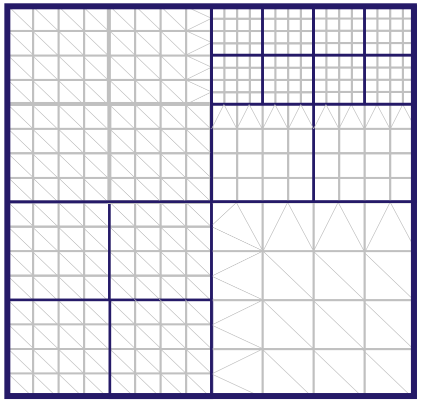
Subdivision Connectivity Grids

Idea:

- Do the same thing (hierarchical triangulation)
- But use a grid of many triangles in each node:



Subdivision Connectivity Grids



Advantage:

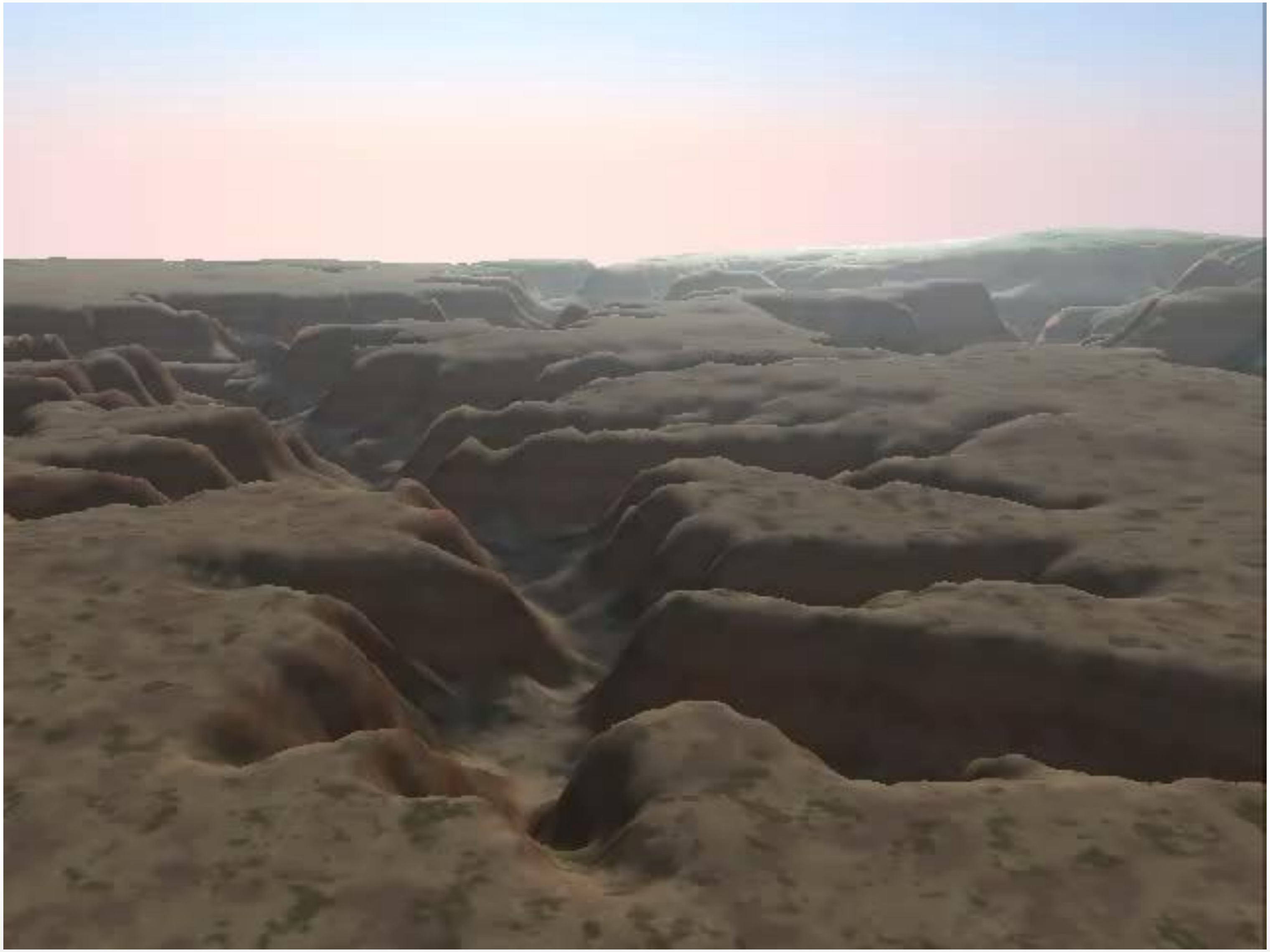
- Amortizes hierarchy creation / traversal costs over many triangles
- Well suited for graphics hardware (GPU) implementations (regular structure)

Disadvantage:

- Less adaptivity
- This is ok for the $1/z$ term in perspective rendering (we will see that later)
- But geometry will be oversampled

Example

Example



Example

Constraint Triangulations

Additional Constraints:

- Vertices, edges, area
- Need to augment subdivision algorithm

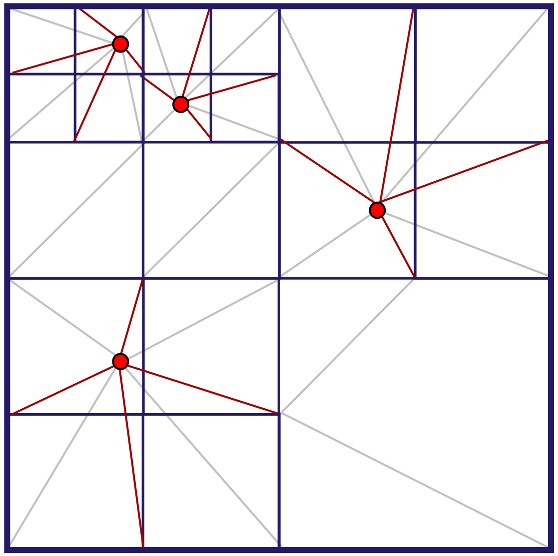
Hierarchical Subdivision:

- Subdivide until a simple case is found
  - At most one vertex in each cell
  - At most one line segment intersecting each cell
  - At most two boundary / cell intersections
- Then triangulate according to fixed rules

Vertex Constraints

Vertex Constraints:

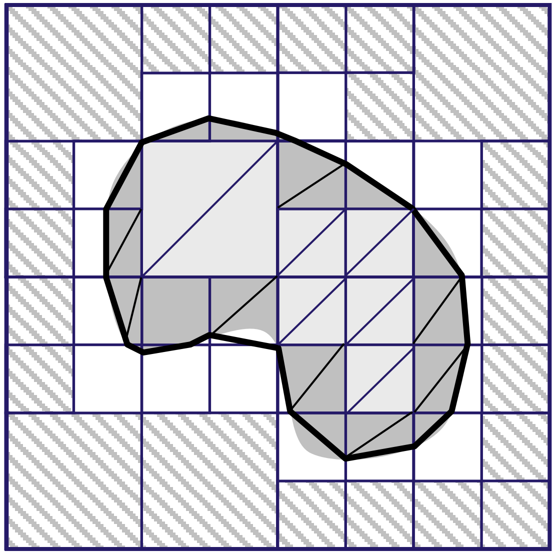
- When only one point is left in each box
- Subdivide once more
- Move center to point
- Then balance and triangulate (proceed as before)



Edge / Area Constraints

Edge and area constraints

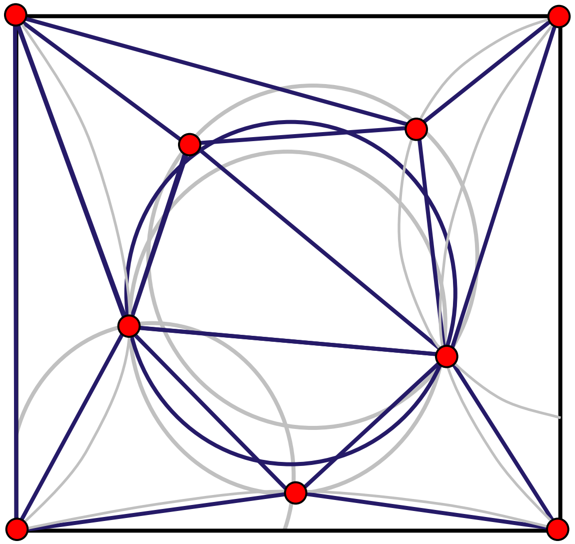
- Subdivide until intersection with edges / boundary curves has constant complexity (e.g. two intersections per cell)
- Then apply fixed subdivision rule
- Edge constraints:
  - Keep all triangles
- Area constraint:
  - Delete outside triangles



Alternative Algorithm

Alternative: (constrained) Delaunay triangulation

- Delaunay triangulation of a point set:
  - Triangulation in which the circumcircle of each triangle is empty
  - This triangulation *maximizes* the *minimum angle* in any triangle
  - The triangulation always exist
  - Can be computed by iterated edge flipping or (more efficiently) by line sweep algorithms (O(n log n) time for n points)



- Constrained Delaunay triangulation:
  - Additional edge / polygonal area constraints
  - More involved to compute

Spatial Data Structures

Range Queries, Collision Detection

Spatial Data Structures

Motivation:

- Common problems:
  - Select a handle point by mouse click (millions of handles)
  - Click on other stuff (edges, triangles, patches)
  - Find the nearest point in a point set
  - Find the *k* nearest points (e.g. for surface fitting)
  - Find all geometry within a range (cube, sphere, etc.)
- This should work on large models
  - Billions of primitives
  - Frequent operations
    - E.g.: compute 20 nearest points for 1.000.000 points
    - Quadratic runtime is unacceptable
- Such operations can be speed up tremendously using spatial indexing data structures

Spatial Data Structures

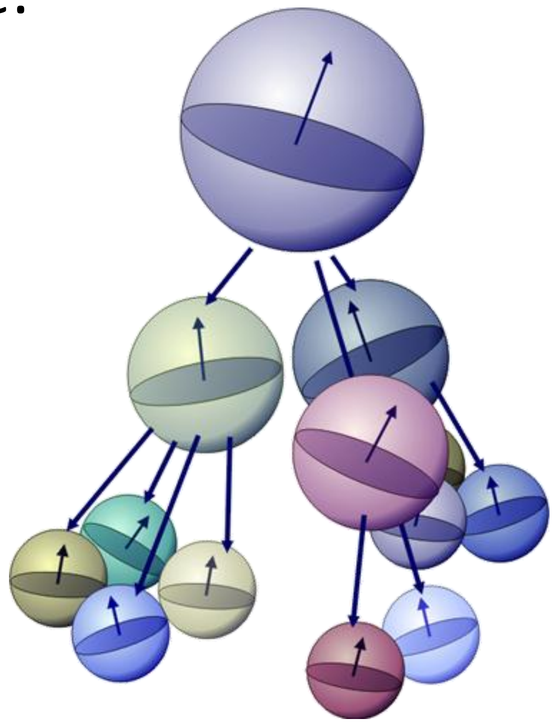
Basic Idea: Hierarchical decomposition of space

- Almost all approaches commonly used in practice are based on hierarchical spatial decompositions
- For some problems, there are more sophisticated data structures from computational geometry, but they often have to large space requirements
- In practice, anything beyond linear space is out of question

Spatial Data Structures

Basic Idea: Hierarchical decomposition of space

- If the number of objects is still too large:
  - Cluster geometry into a small number of spatially coherent groups
  - Compute a simple bounding volume for each group
  - Apply this principle recursively to all subgroups formed
- We obtain a tree of bounding volumes



Hierarchical Space Partitioning

Formally:

- We have a set of objects $\Omega = \{s_1, ..., s_n\}$, $s_i \subseteq \mathbb{R}^d$ (where d is small, usually $d = 2..3$)
- We form a hierarchy of nodes N_i.
  - Let $C(N_i)$ be the set of child nodes, ...
  - ...and $P(N_i)$ the unique parent node, or *null*, if N_i is the root node R.
- We associate a set of objects $S(N_i)$ with each node N_i.
- We demand $S(R) = \Omega$ (root contains everything) and $N_j \in C(N_i) \Rightarrow S(N_j) \subseteq S(N_i)$ (inner nodes represent the whole subtree)

Hierarchical Space Partitioning

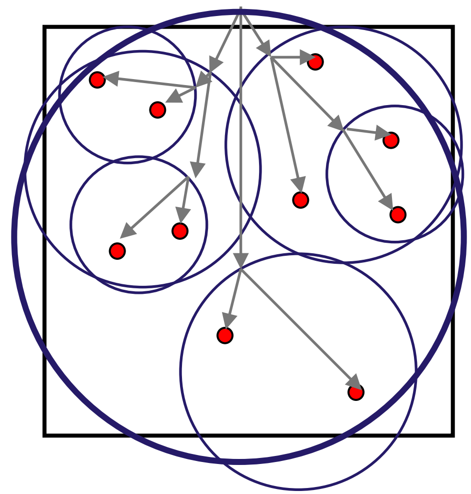
Formally:

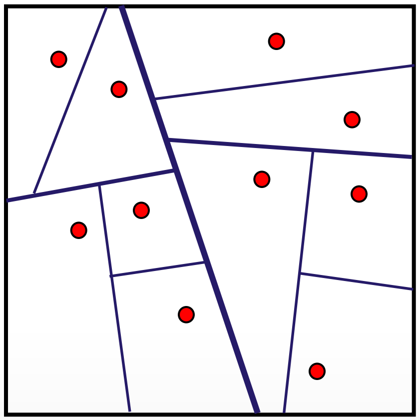
- Bounding volumes: let B(N_i) be a bounding volume of node N_i, B(N_i) $\subseteq \mathbb{R}^d$.
- This means: S(N_i) $\subseteq$ B(N_i)
  (objects are contained in the bounding volume)
- Typically, a bounding volume is a much simpler object than the stored geometry S(N_i).
  - It should be easy to test for intersections with other bounding volumes, geometric ranges and objects to be sorted into the hierarchy.
  - Usually, the memory footprint of B(N_i) is O(1).
  - Axis aligned boxes, spheres and the similar are popular.

Variants

Variants:

- Bounding volume hierarchy
  - Most general definition, we can use any bounding volumes
  - Each inner node represents the union of objects in the subtrees
- BSP-tree
  - Use planes to split the nodes into half-spaces
  - Usually stored as a binary tree (“binary space partition”)
  - Cells are not O(1), but each tree level cuts of a half space, which can be tested incrementally.

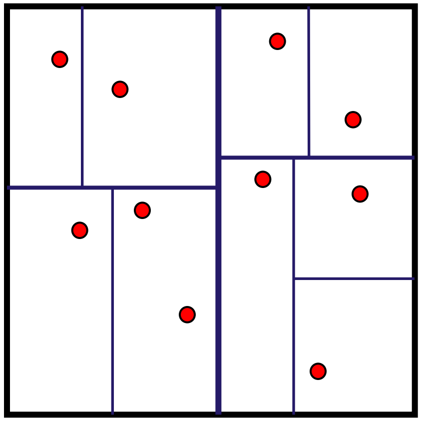


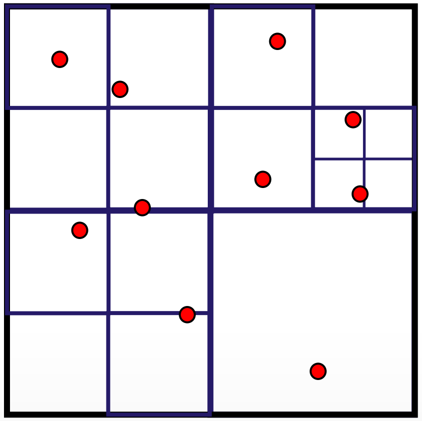


Variants

Variants

- kD-tree / axis aligned BSP tree
  - Use axis parallel splitting planes
  - Special case kD-tree:
    - Cyclically alternating splitting dimensions
    - Use median cut
- Quadtrees / Octrees
  - Always divide into 4 (8) cubes of the same size
  - This is a special case of a BSP- / kD-tree (identifying 3 consecutive binary splits with one octree node)





Extended Objects

Construction for extended objects (other than points)

- Extended objects:
  - Triangles
  - Polygons
  - Patches
  - Line segments
  - etc...
- Division of space might intersect with object
- Two solutions
  - Splitting objects
  - Overlapping nodes

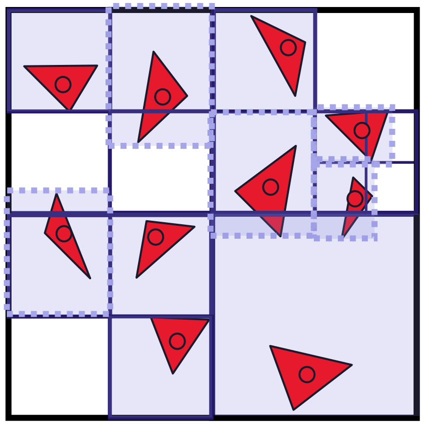
Splitting Objects

First solution: splitting objects

- For example, sorting triangles into a BSP tree:
  - Split each triangle along splitting plane, if necessary
  - Try to optimize such that as few as possible triangles are split
- (Rather) easy to see:
  - A BSP tree needs at least worst case $O(n^2)$ fragments for n triangles (in practice typically $\approx O(n \log n)$)
  - This is worst-case quadratic storage
  - The same bound also applies to kD trees, octrees etc (special cases)
- Splitting objects is usually too expensive
  - Used in early low-polygon 3D engines for visibility computation

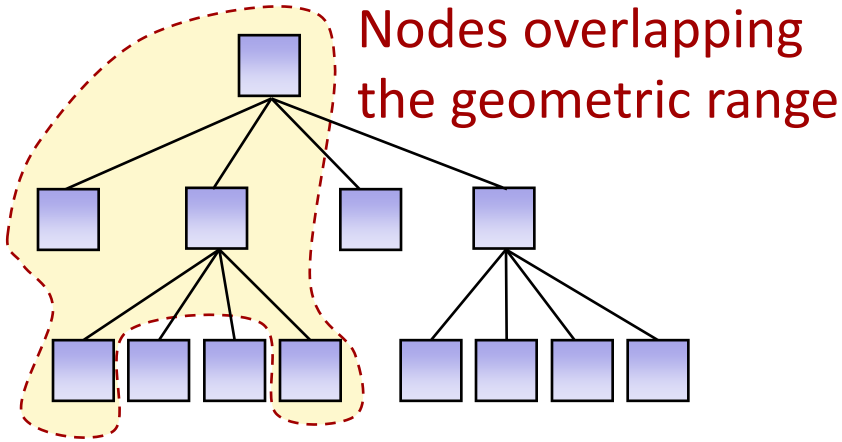
Overlapping Regions

Other alternative:



- Allow objects to exceed the region associated with each node
- Store a second, extended bounding box to reflect this information
- Typical strategy:
  - Allow up to 10% oversize (exceeding node limits by 10% in each direction)
  - If this does not fit into leaf nodes, use an inner node.
- Effective bounding volumes may overlap now
  - Limiting the percentage limits the amount of space covered multiple times (e.g. 10% in each direction means $1.2^3 \approx 1.7\times$)

Range Query Algorithm

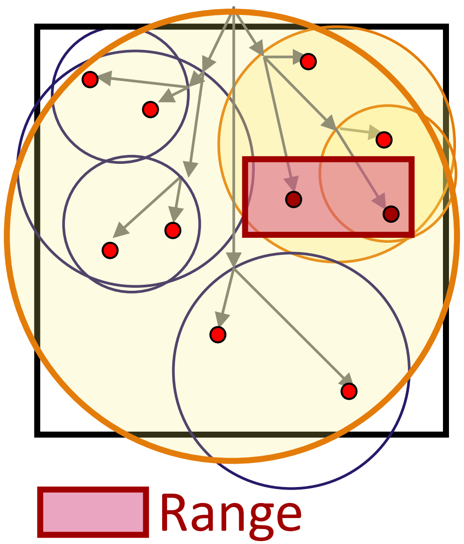


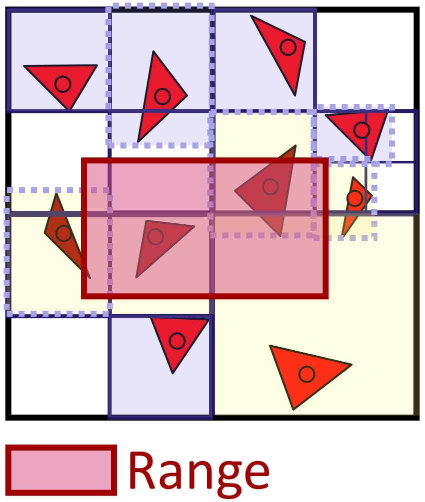
Start at root node: Then, recursively

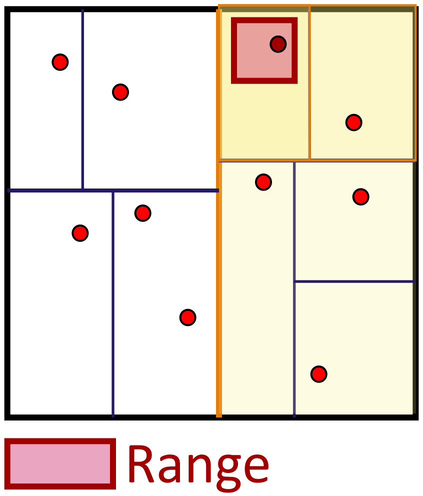
- If range overlaps bounding box
  - Collect inner node primitives
  - Test for range intersection
  - Go on recursively for child nodes
- If range does not overlap bounding box
  - End recursion

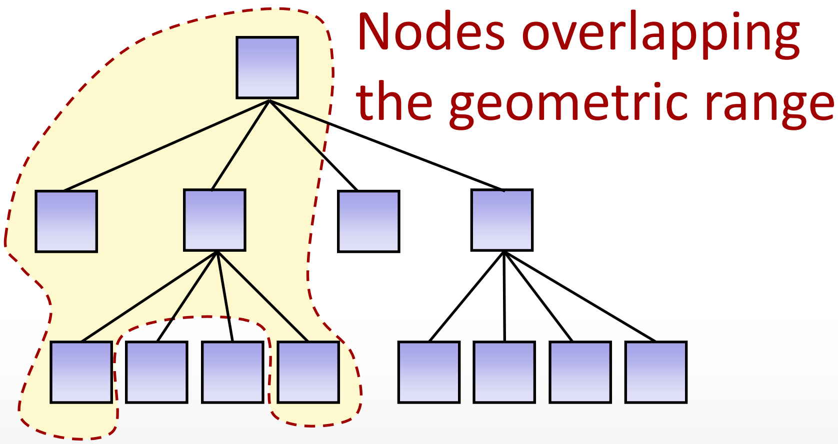
works for all
hierarchy types

Examples









Parametric Surfaces

In case every primitive itself is a parametric object:

- We can “continue” the hierarchy
- Use a regular subdivision of the parameter domain (binary splits, quadtree)
- Form bounding volumes dynamically (e.g. convex hull of subdivided control points)

Collision Detection

Related Problem: Collision Detection

- We want to compute whether two geometric objects intersect with each other
- Important problem for dynamic simulations
- Also useful for CAD applications (arrange objects that do not collide)

Simple Solution:

- Test every part of object A for collision with every part of object B (e.g. each triangle with each other triangle)
- This is usually to expensive [O(mn)]

Hierarchical Collision Detection

Hierarchical Collision Detection

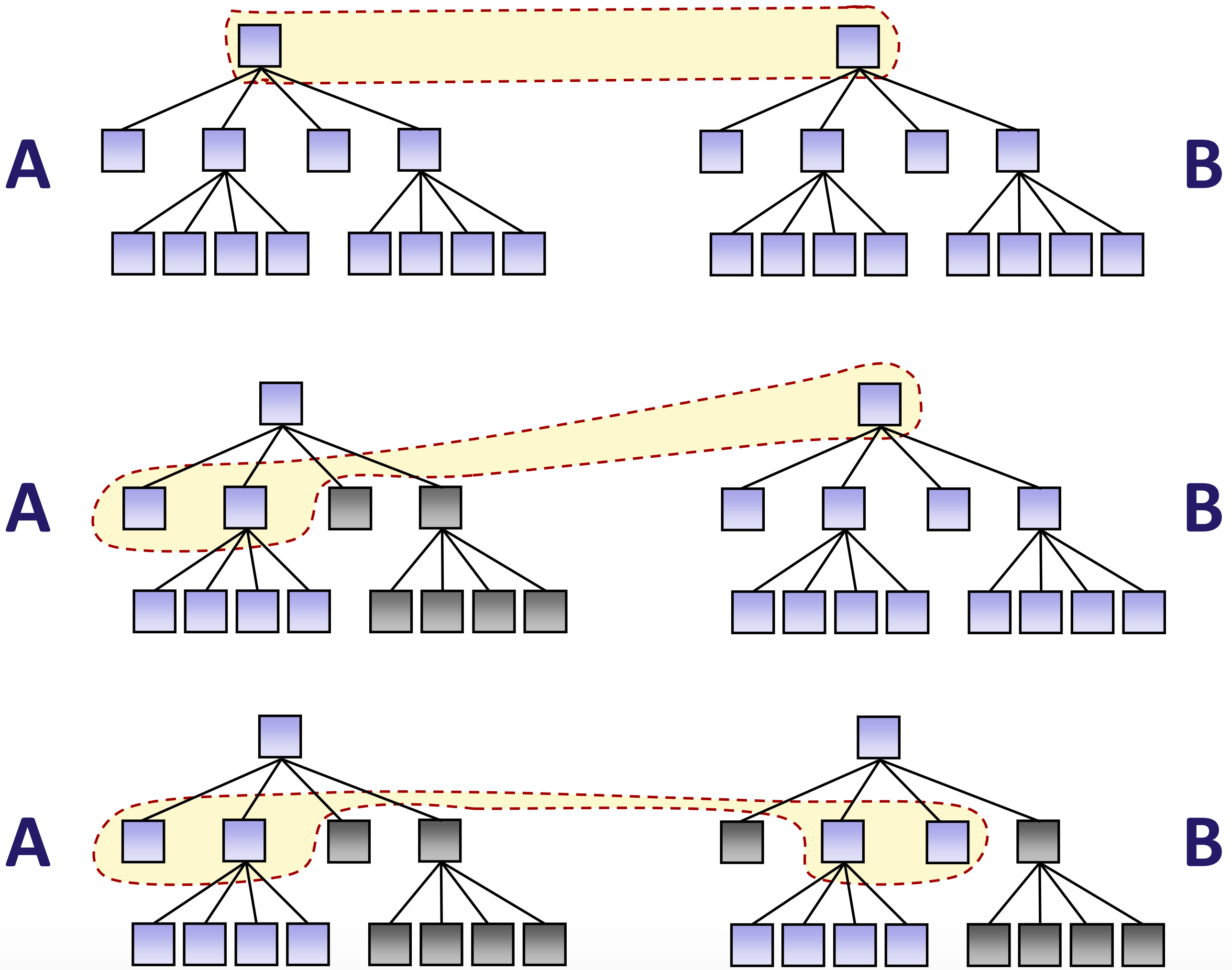
- Precompute a hierarchy for both objects *A* and *B* that should be tested for collision.
- Then apply a hierarchical collision test (next slide)

Hierarchical Collision Test

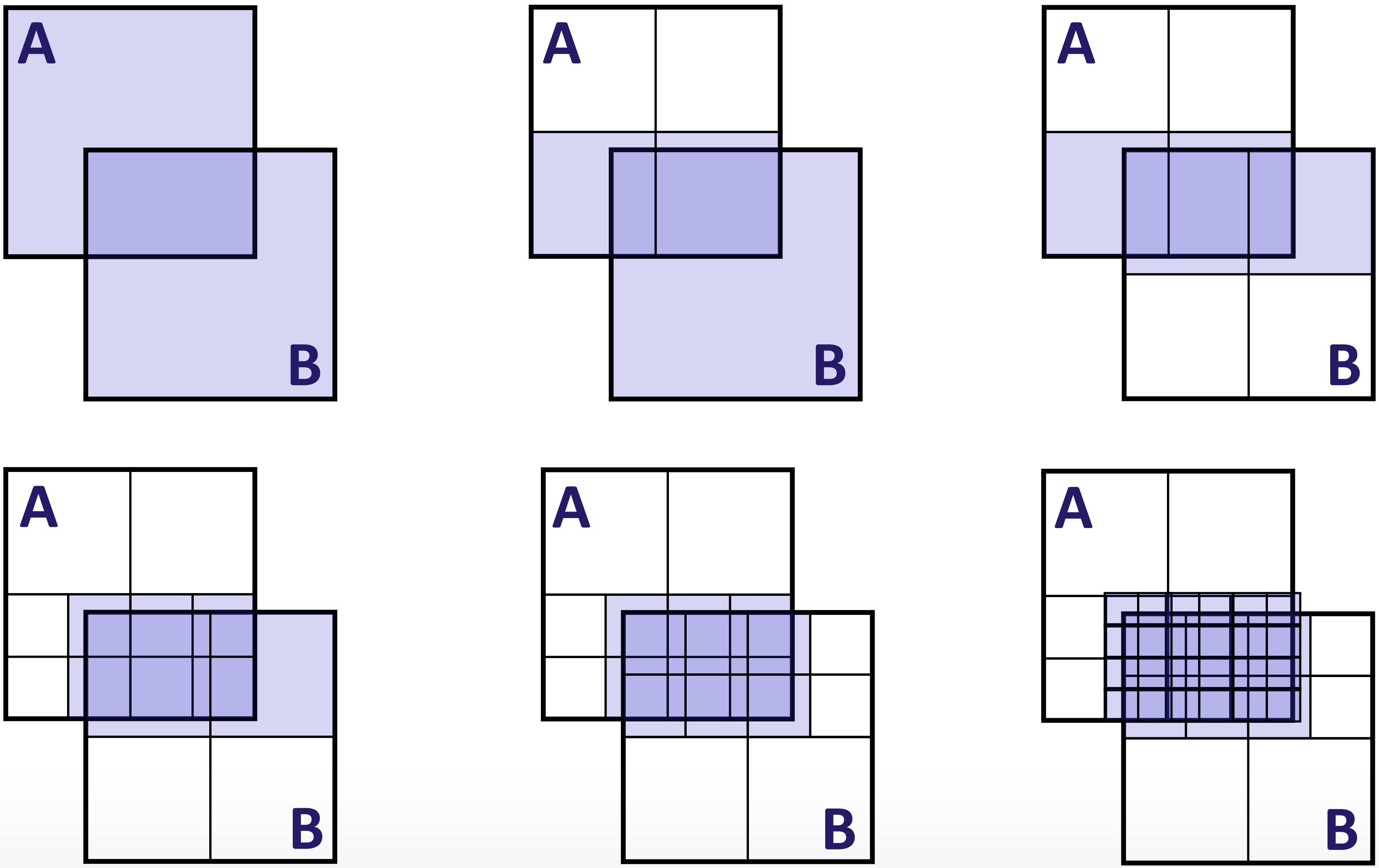
Collision Test: Input – nodes N_A, N_B from objects A, B.

- Test bounding volumes B(N_A), B(N_B) for intersection
- **If** $B(N_A) \cap B(N_B) \neq \emptyset$**:**
  - Test all objects S(N_A), S(N_B) for intersection
  - Output those objects that do intersect
  - **If** diameter(B(N_A)) > diameter(B(N_B)):
    - For all children $C \in C(N_A)$
      - CollisionTest(C, N_B)
  - **Otherwise:**
    - For all children $C \in C(N_B)$
      - CollisionTest(C, N_A)

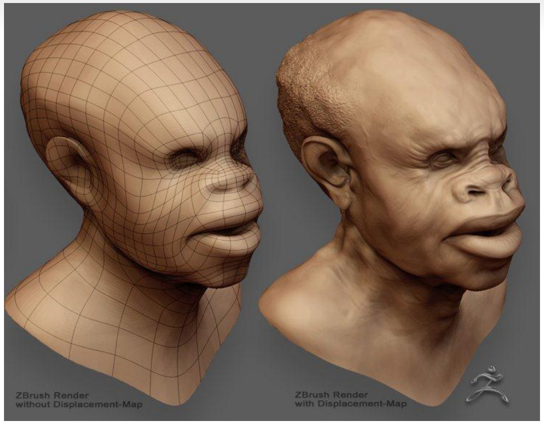
Illustration



Illustration



Ray-Heightfield Intersections



- Collision detection
- Effect of a highly tesselated mesh
- Used in games and scientific visualizations
- Very handy tool for geometric modelling

Maximum Mipmaps

Equivalent to fully sub-divided quad-tree [Samet 1990]

Developed in our group in 2008

Already used in a some of cg publications

- soft shadow rendering [Guennebaud 2006]
- geometry image intersection [Carr et al. 2006]

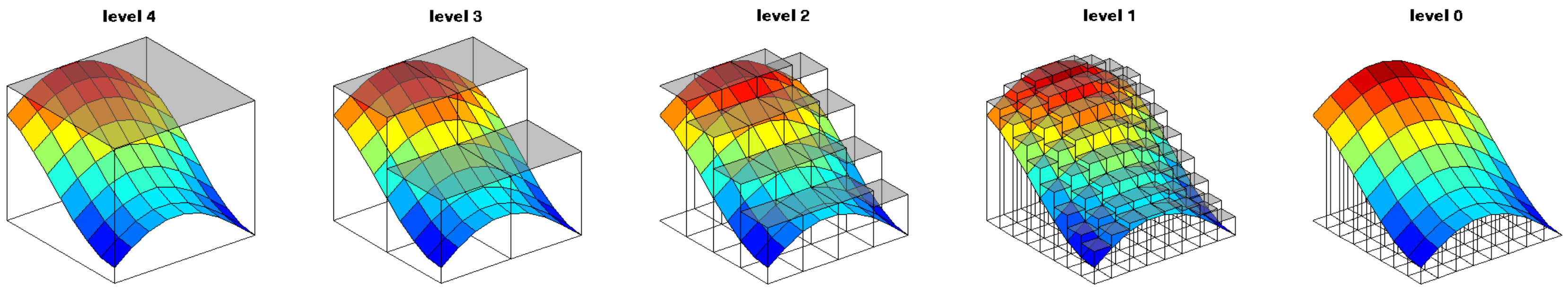
MMM Datastructure is dynamic

- precomputation time in order of ms

4/3 amount of additional memory required

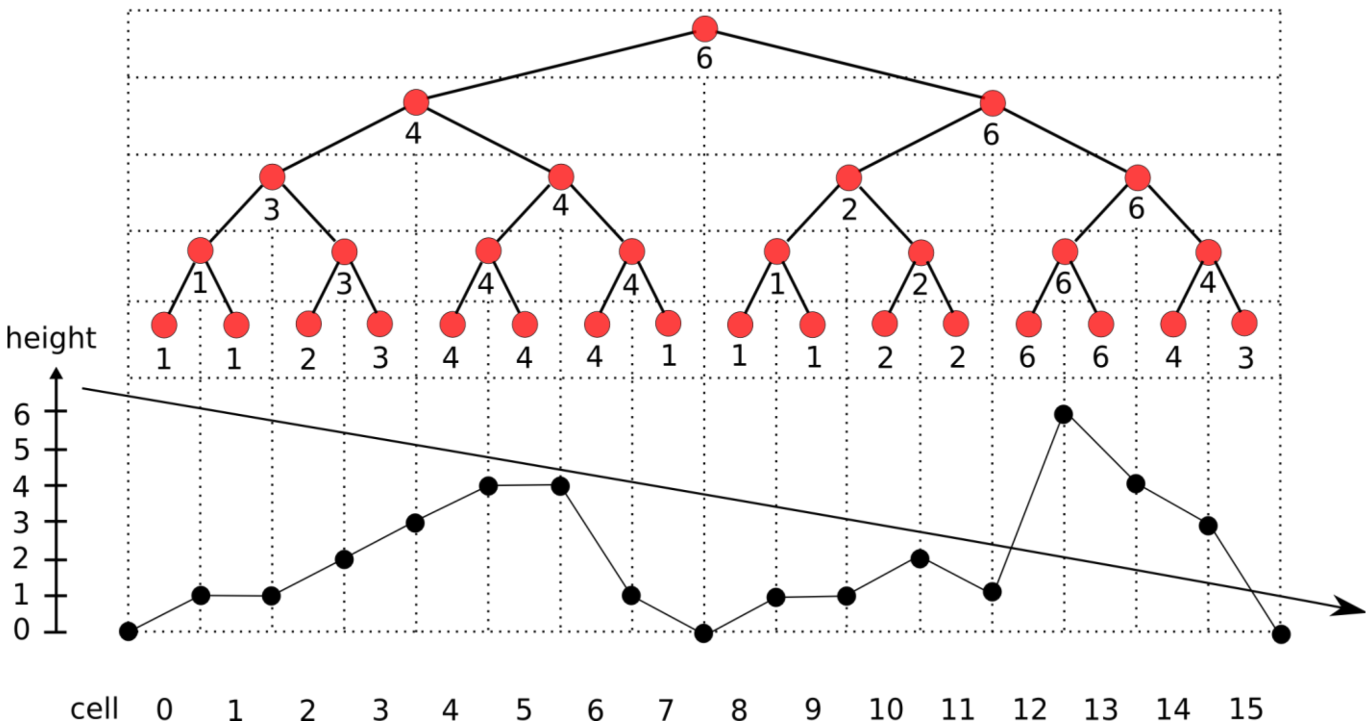
Real-time rendering

Maximum Mipmaps



- Collection of bilinear patches placed on a regular grid
- Level 1 to n – maximum height of underlying patches
- Level 0 – vec4 (RGBA) value storing height of the bilinear patch data points
- due to optimized hardware the construction time is incredibly fast

Intersection Algorithm

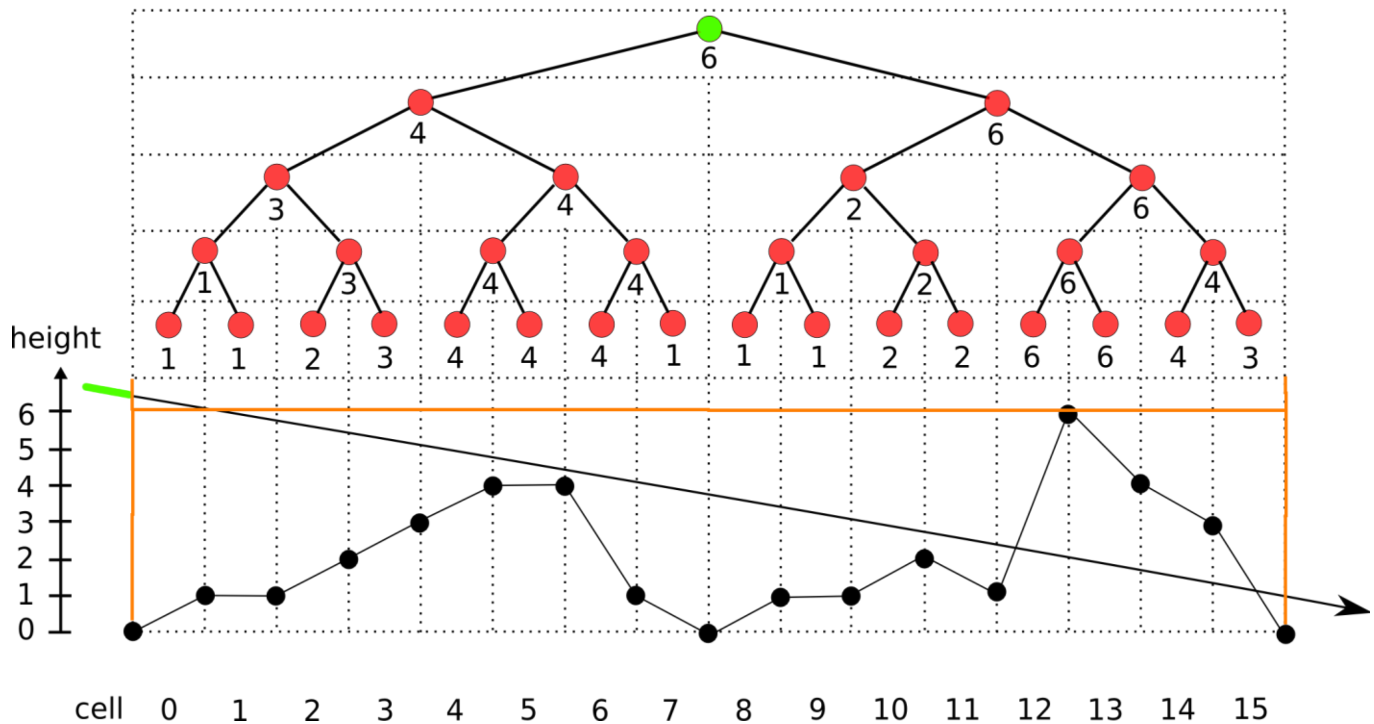


Example of a Ray – Height Field Intersection

1D heightfield and the corresponding MMM datastructure

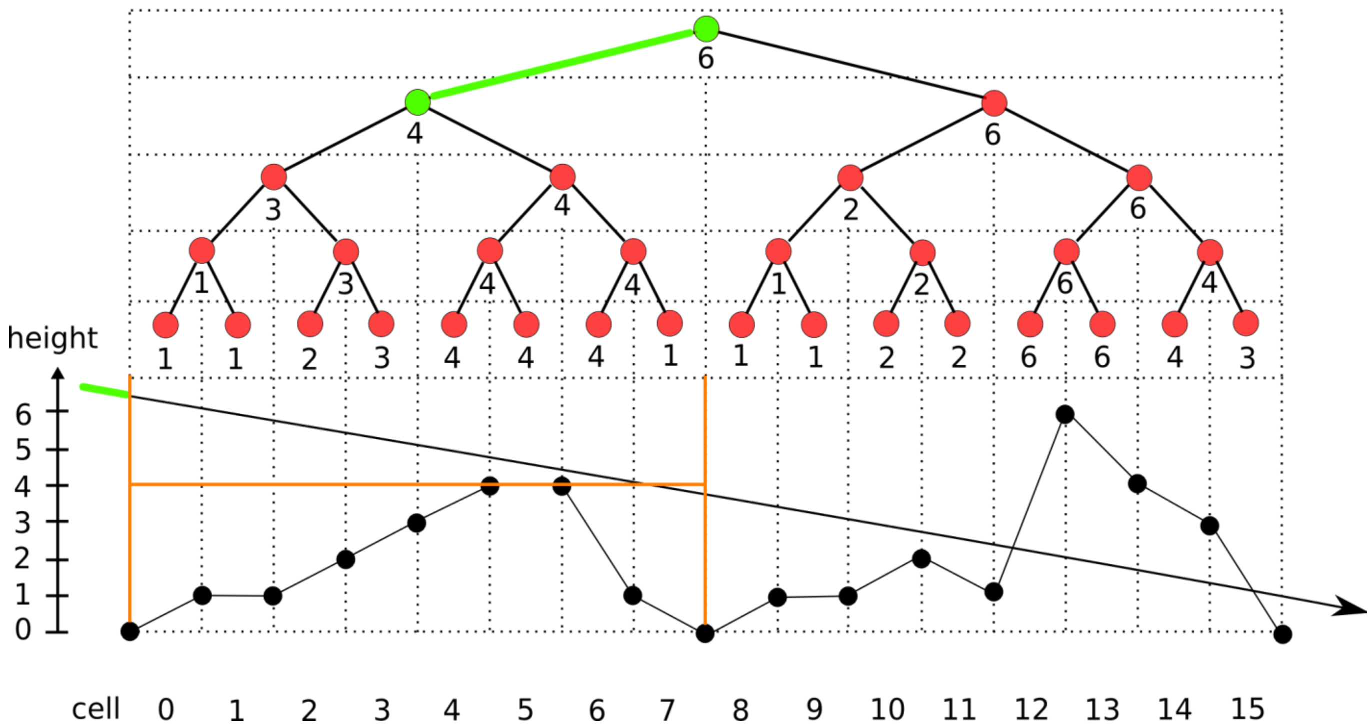
- linear elements in the finest levels

Intersection Algorithm



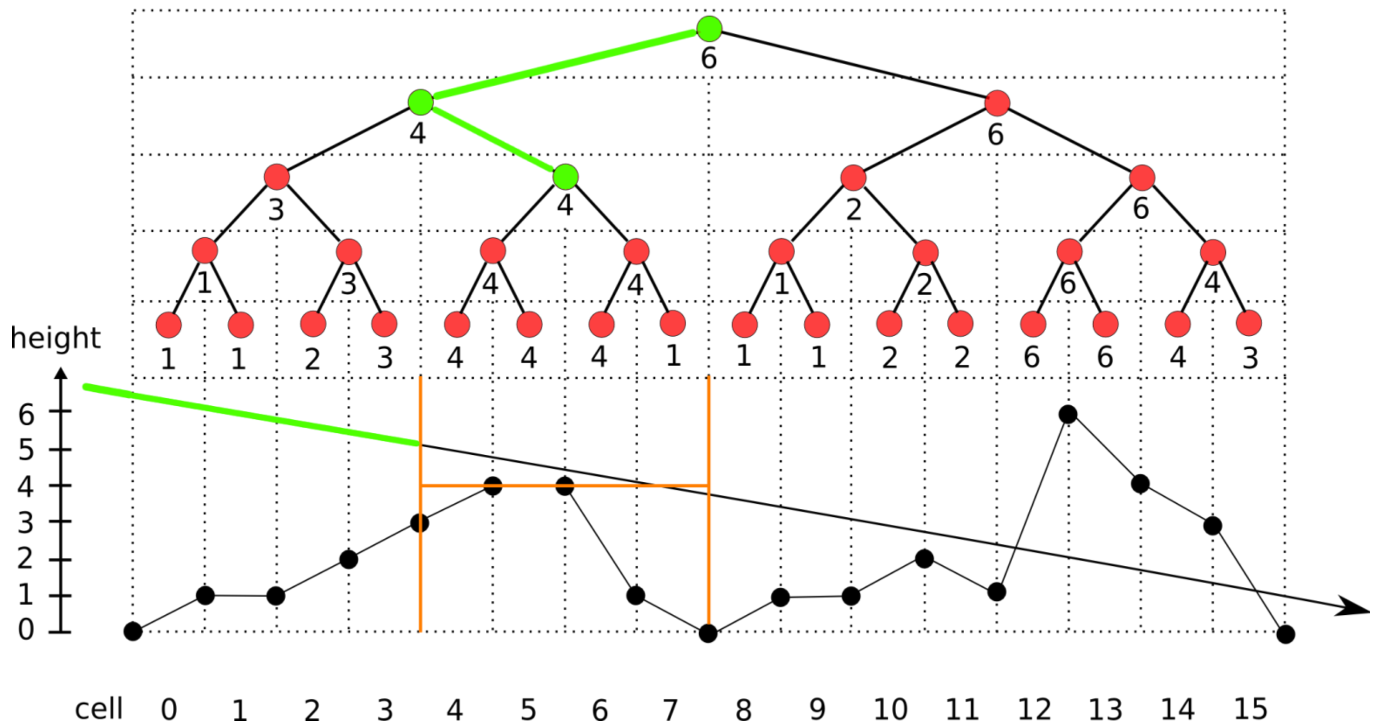
Ray hits the bounding box of the Height Field

Intersection Algorithm



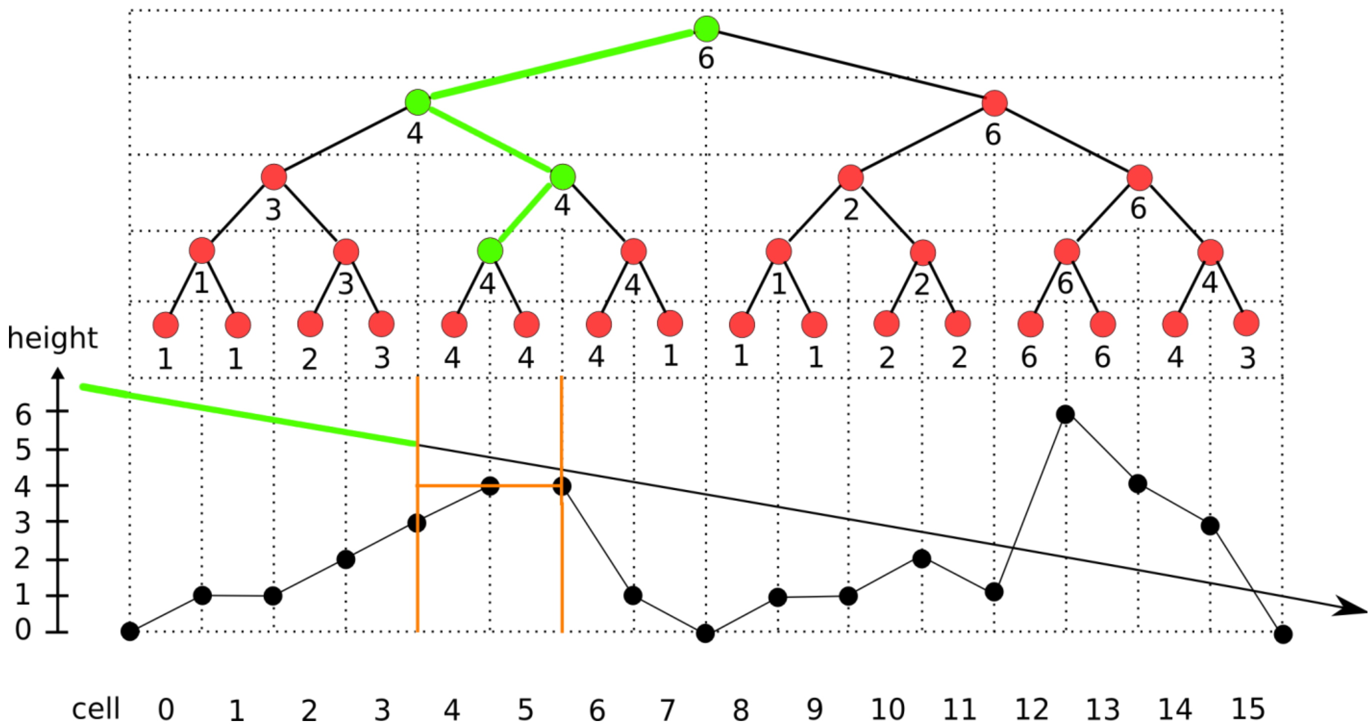
Traverse down the mipmap tree, since the ray hits the maximum plane of the current cell

Intersection Algorithm



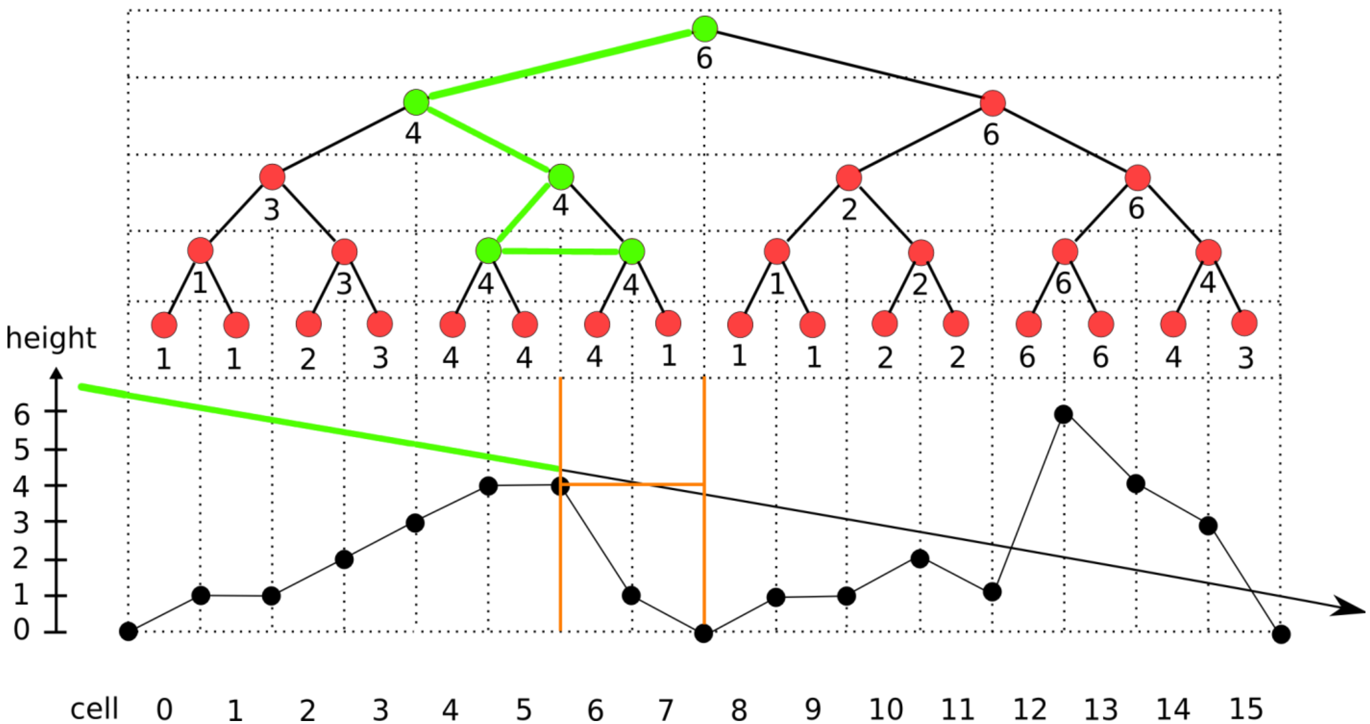
Move the ray to the next boundary, since it does not hit the maximum height plane of the current cell

Intersection Algorithm



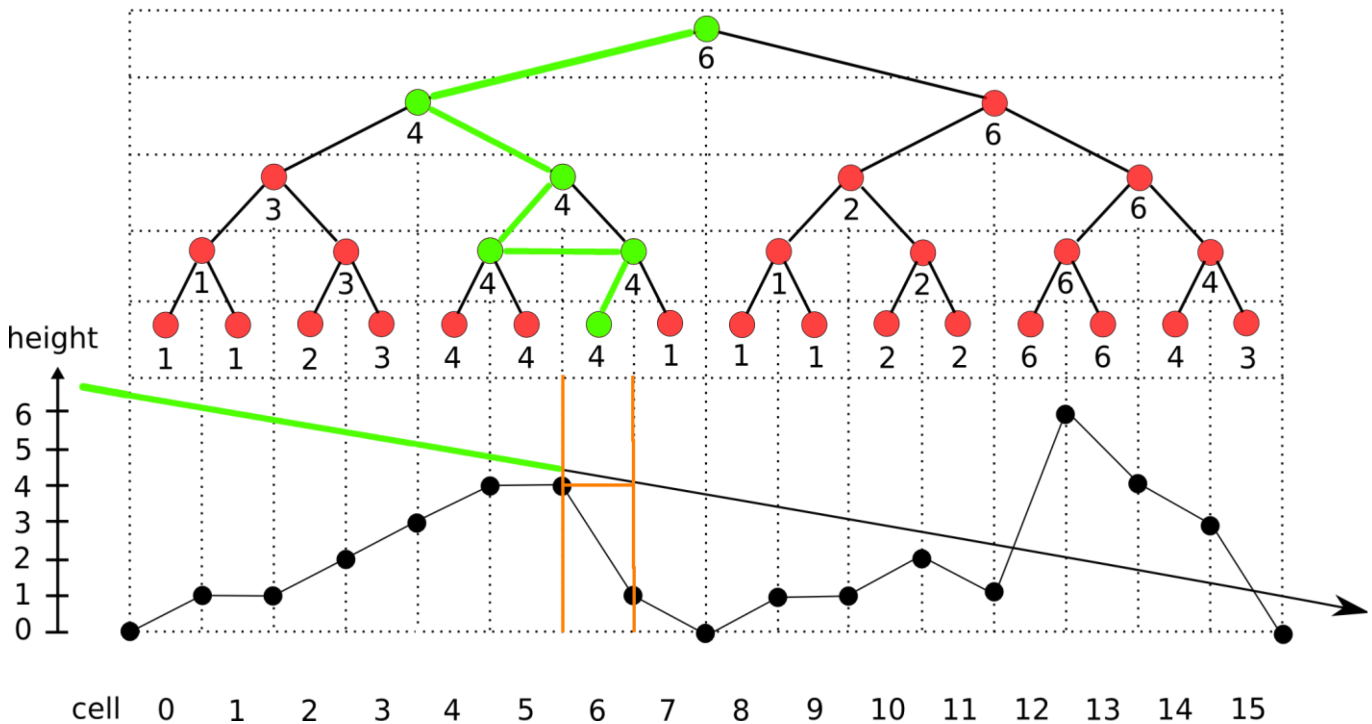
Traverse down the tree

Intersection Algorithm



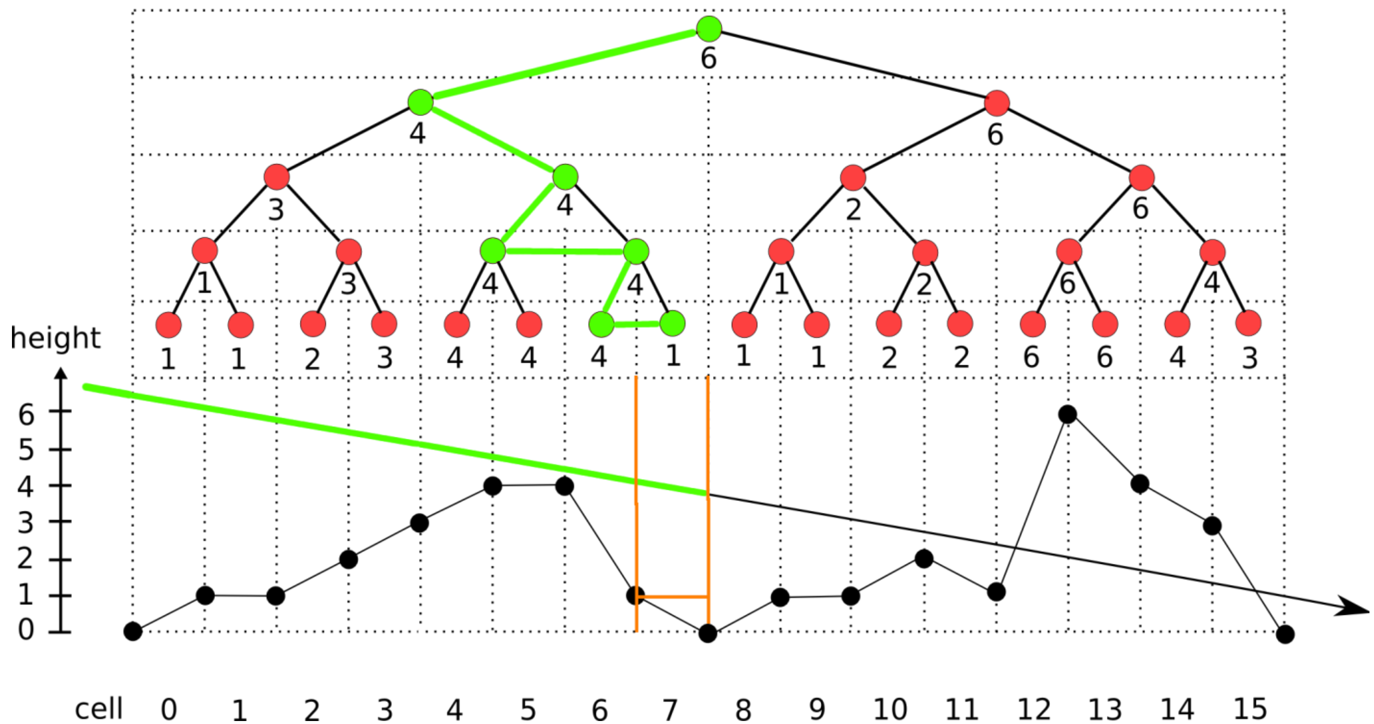
Go to next sibling node, since the ray doesn't intersect the maximal height plane of the cell

Intersection Algorithm



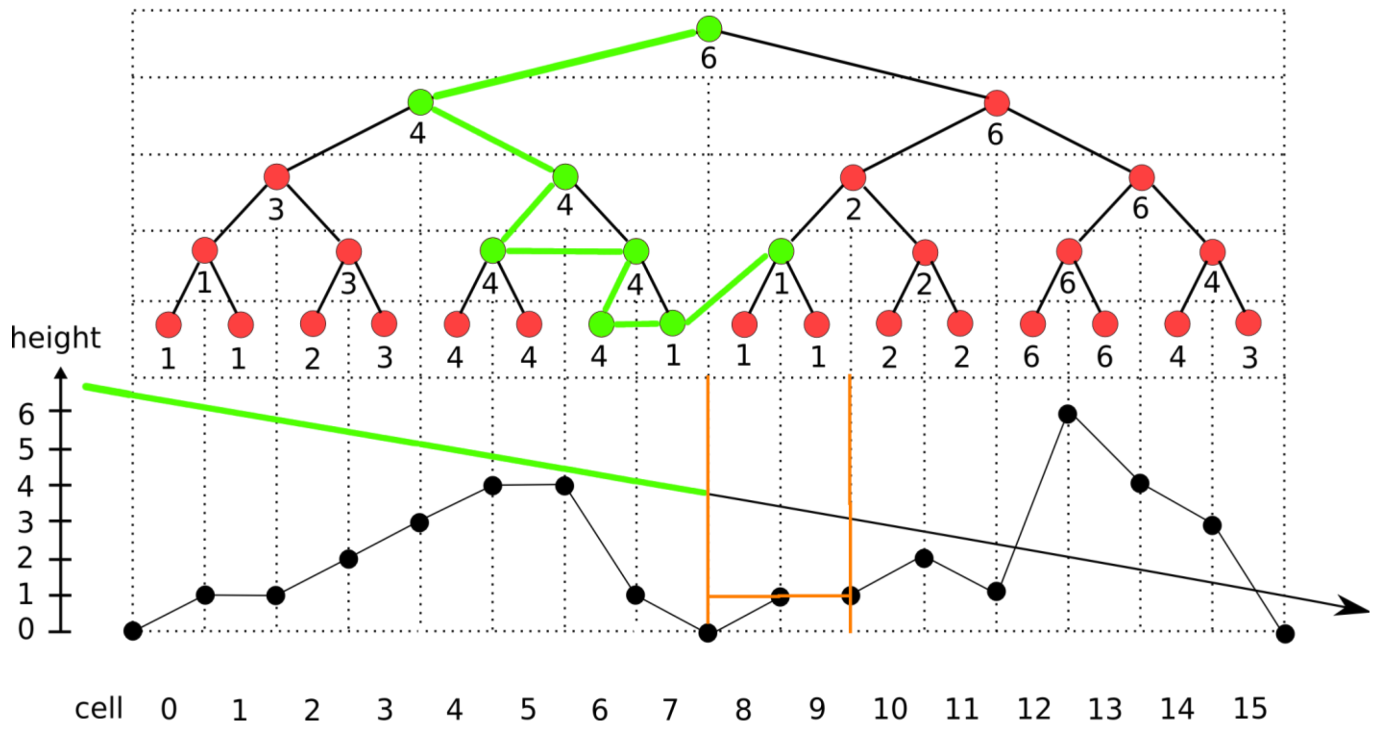
traverse down

Intersection Algorithm



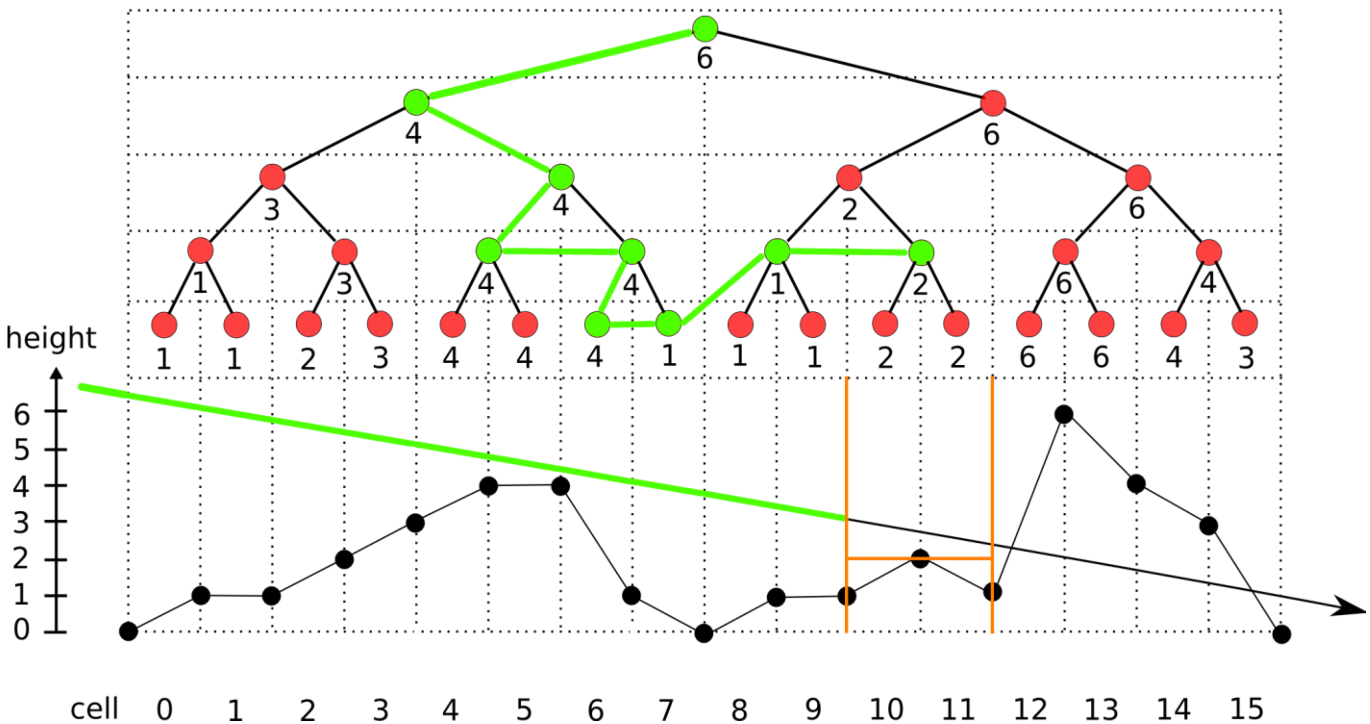
move ray to the boundary

Intersection Algorithm



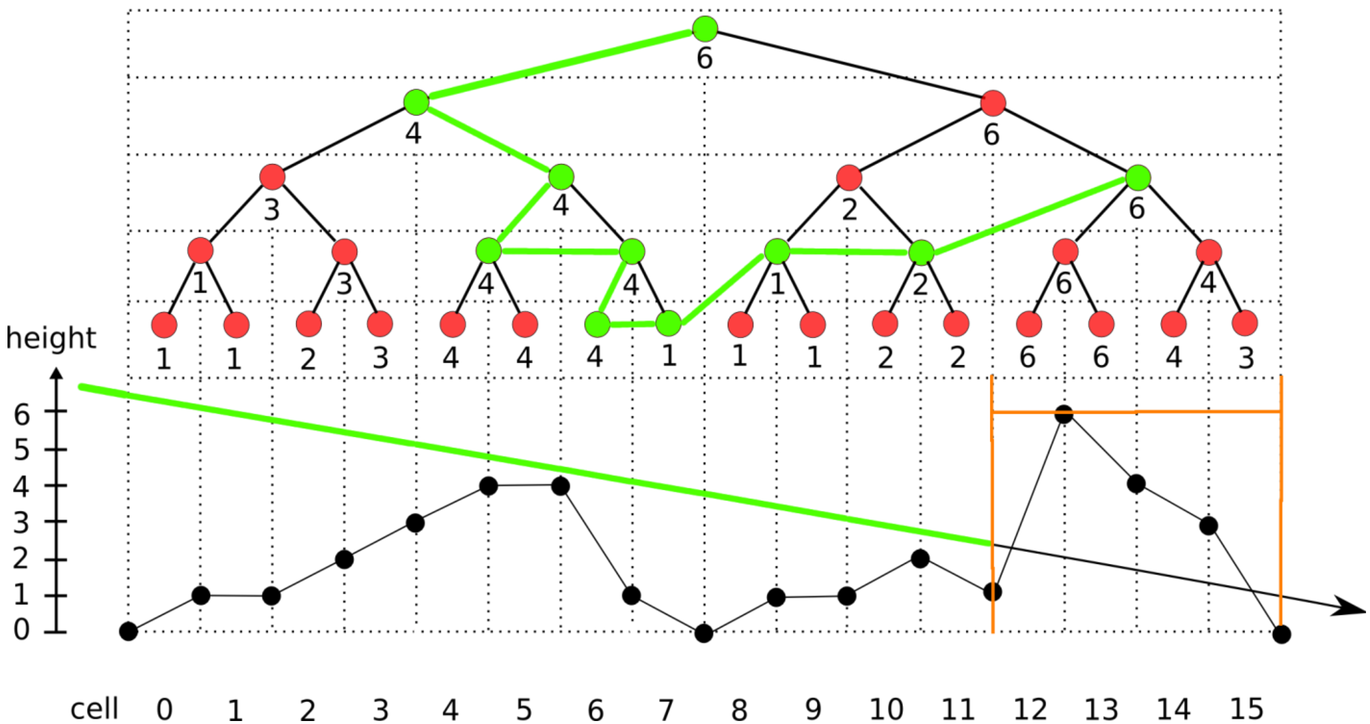
ray at cell boundary divisible by two, hence increase the mipmap level (traverse up in the tree)

Intersection Algorithm



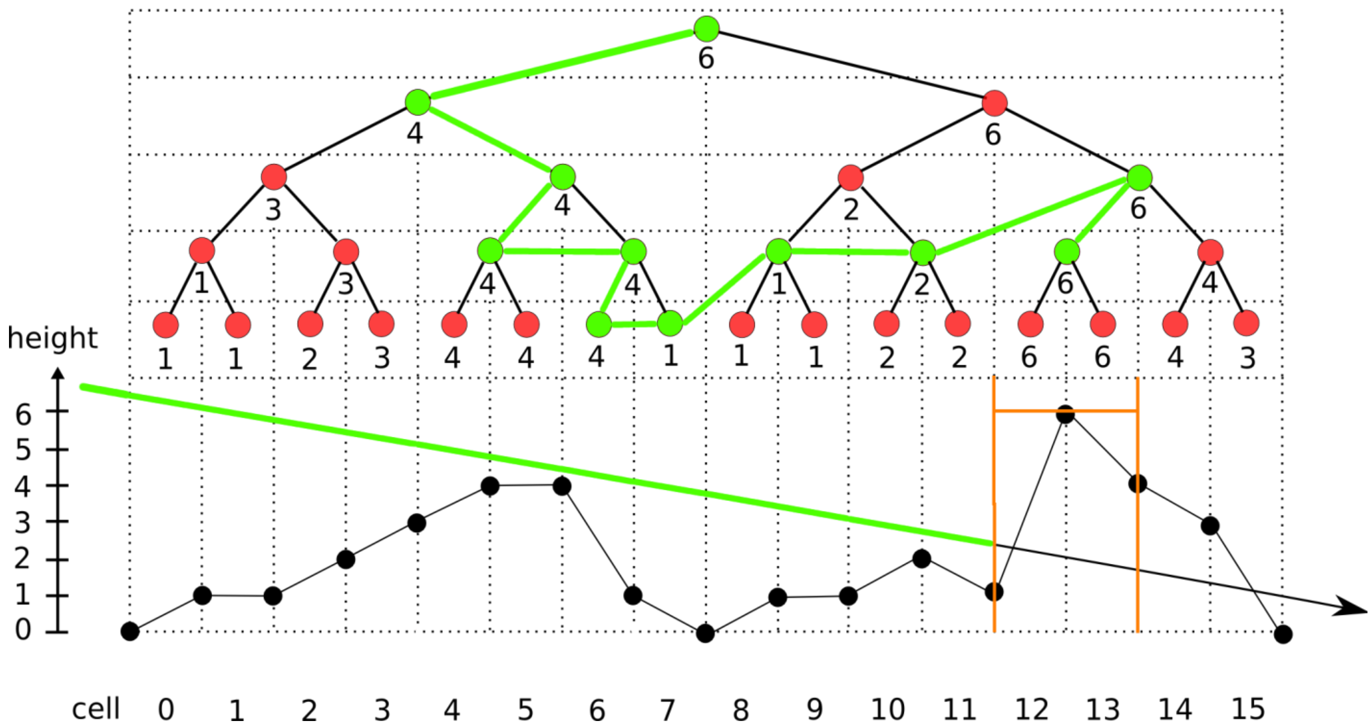
move ray to the boundary

Intersection Algorithm



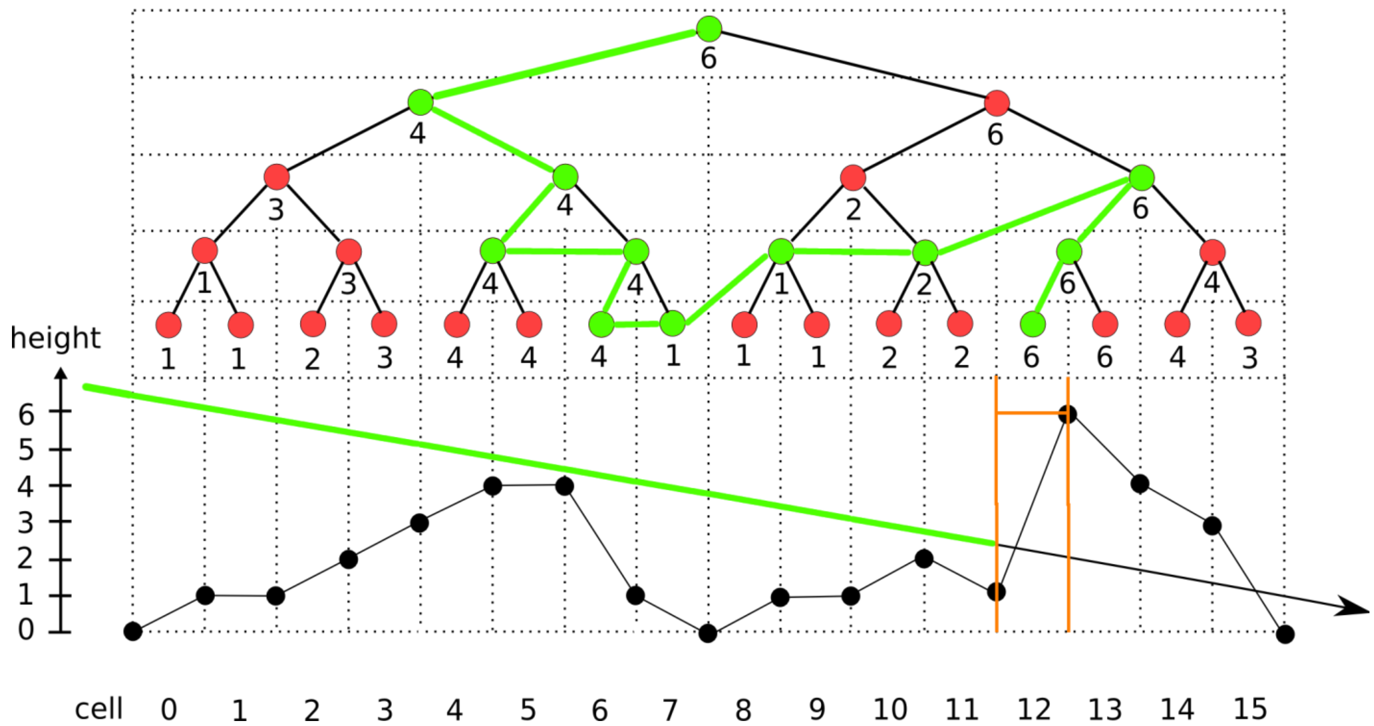
ray at boundary divisible by two, hence increase the level

Intersection Algorithm



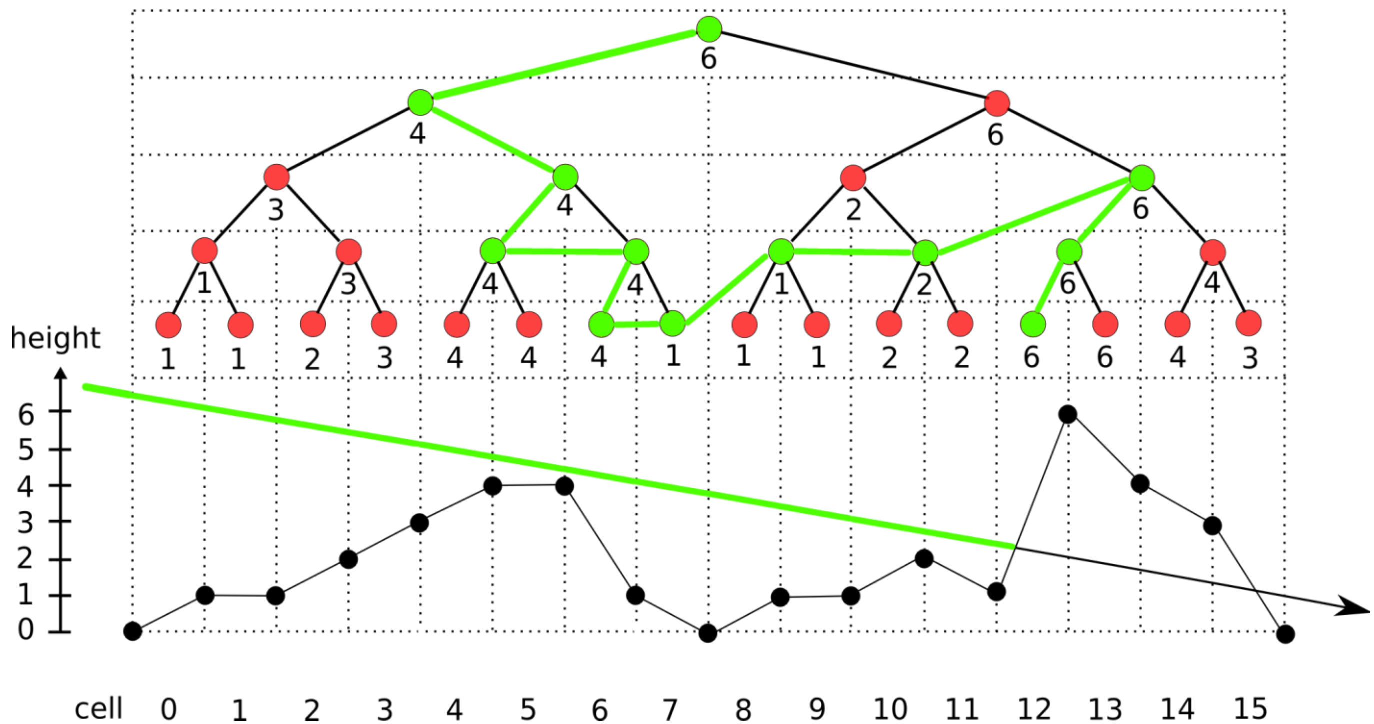
ray below the maximum height, hence decrease the level

Intersection Algorithm



ray below the maximum height, hence decrease the level

Intersection Algorithm



level = 0, hence perform ray-line intersection test

Ray – Height Field Intersection point is found

Heightfield rendering

Video

Parametric Objects

Collision of parametric objects:

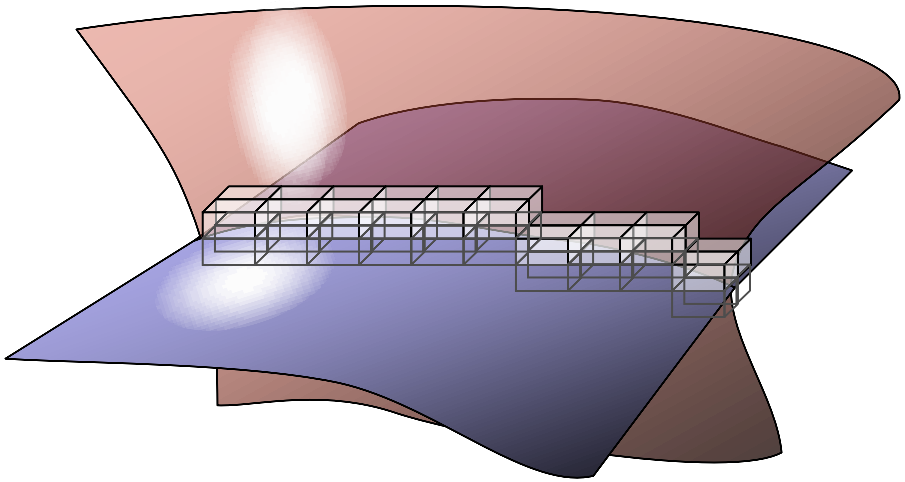
- Again, we can “continue” the hierarchy in the parametric domain
- Useful for speeding up patch-patch collision detection
- We can also compute intersection lines hierarchically

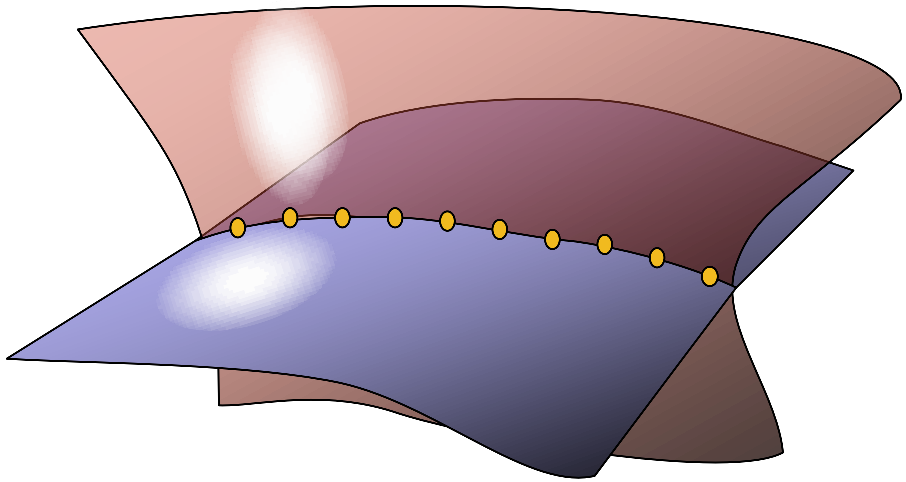
Parametric Objects

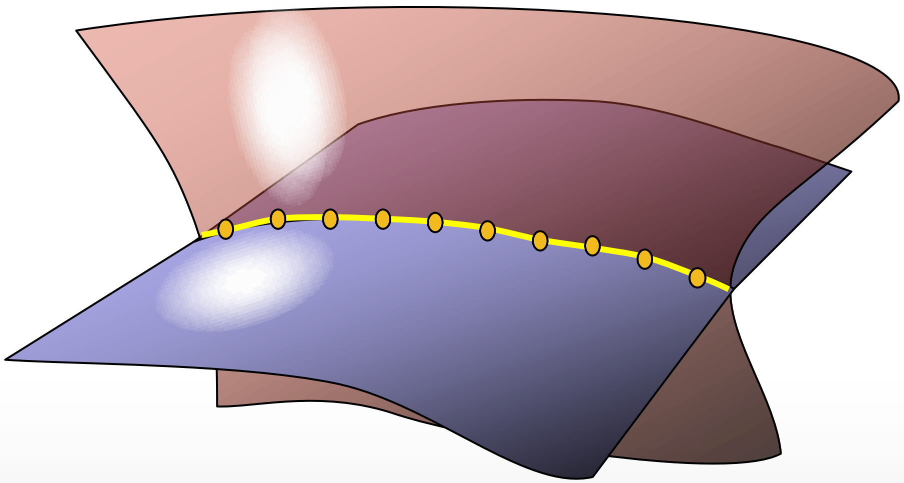
Computing intersection lines:

- Hierarchical intersections until a number of small boxes is left
- Place a control point in each box
- Use a Newton iteration to project points on intersection line
  - Move points in direction orthogonal to line only (avoid degeneracies)
- Fit a spline through the control points (spline interpolation problem, linear system)
- Can be additionally constrained to lie on intersection line
  - Minimize integral residual of distances to patches
  - But this is a non-linear optimization problem (Newton solver)

Intersection lines

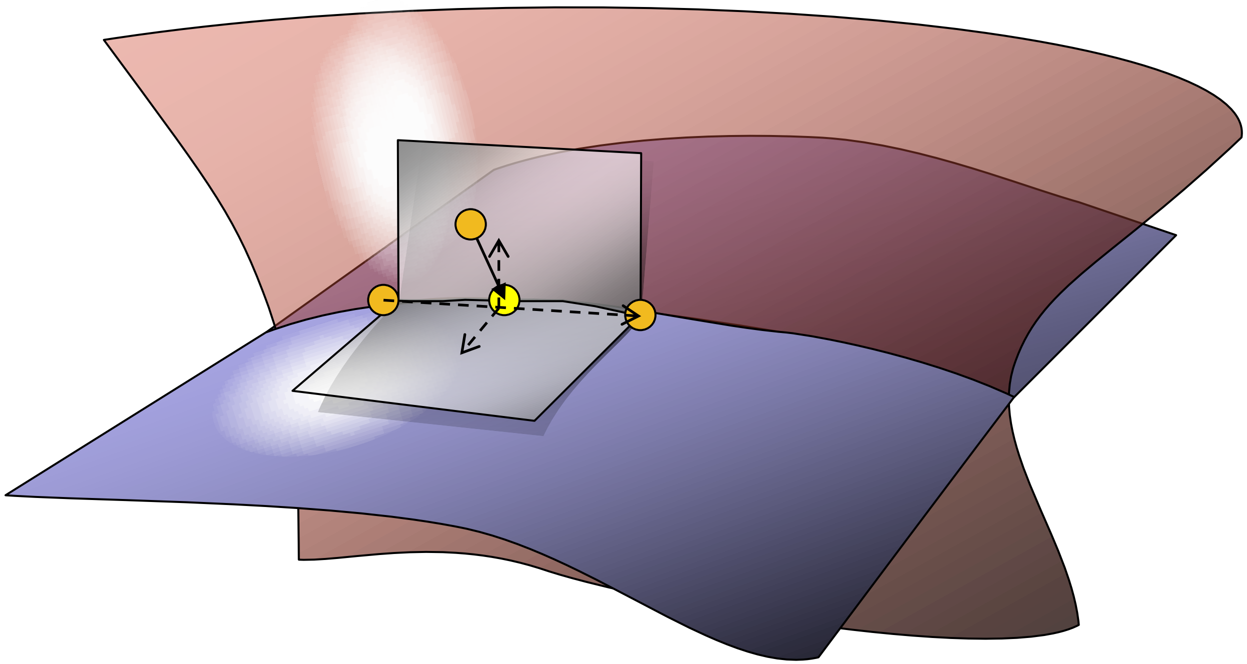






Projecting a Point

Quasi-Newton Scheme



Nearest Neighbor Queries

Problem:

- Given n objects s_i and a point **p** in space
- Two variants:
  - Find the object that is closest to **p**
  - Find the k closest objects (k-nearest neighbors, kNN)

Operations:

- Compute distance point $\leftrightarrow$ primitive
- Compute distance point $\leftrightarrow$ bounding volume

Hierarchical Query Algorithm

Data Structures:

- The query algorithm needs some bounding volume hierarchy for the objects
  - A kD tree works best in practice, but other data structures also do the job
- In addition, two auxiliary data structures are needed:
  - A priority queue of objects Q_{obj}
  - A priority queue of bounding volumes Q_{BB}
  - Both sorted by distance to the query point

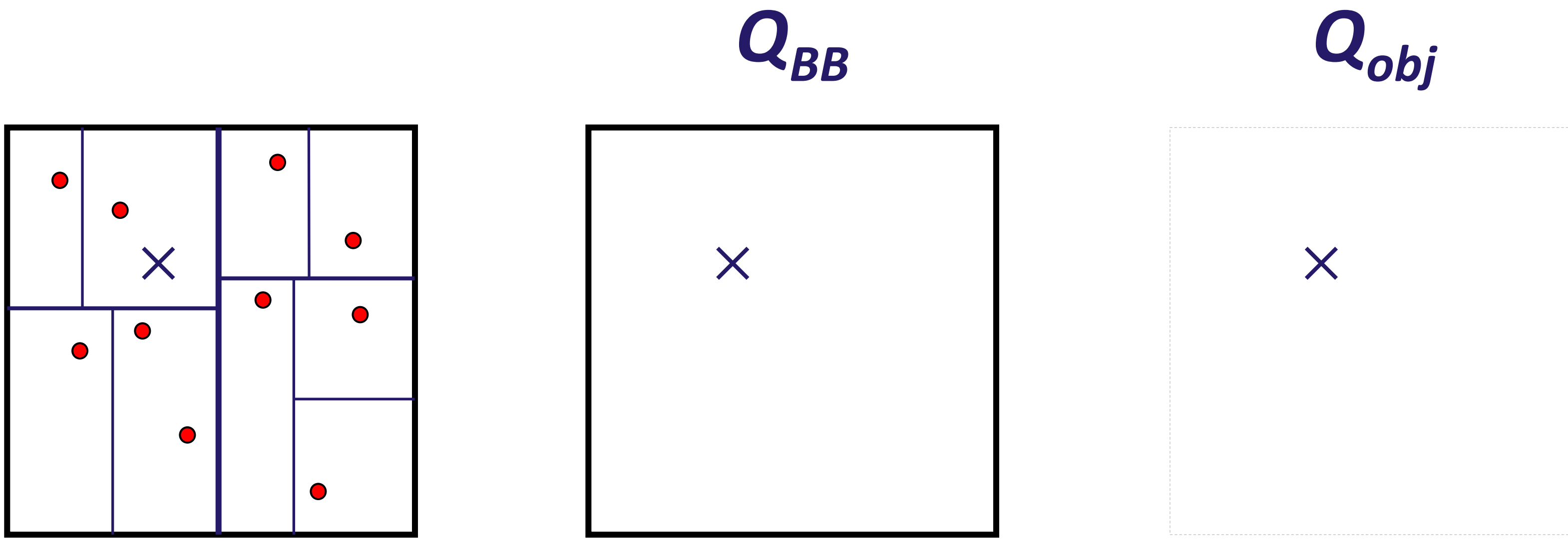
Hierarchical Query Algorithm

Algorithm: Compute k nearest neighbors

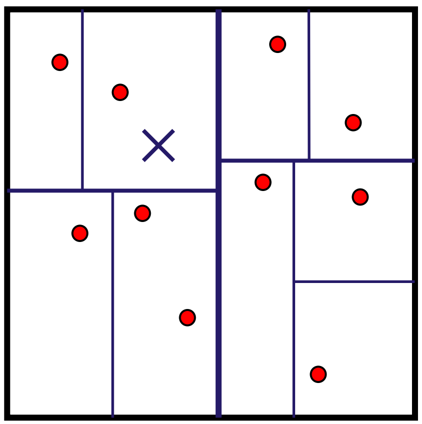
Input: Hierarchy of objects N, query point **p**

- **Initialization:** Put root node on Q_{BB}
- **While** *#output* < k **and** both priority queues non-empty
  - Compute distance to min(Q_{BB}) and min(Q_{obj})
  - **If** an object is closer
    - output the object
  - **Otherwise, if** a box is closer
    - Take the box from the queue
    - Insert all objects into Q_{obj} and all child nodes into Q_{BB} (for this, the corresponding distances need to be computed)

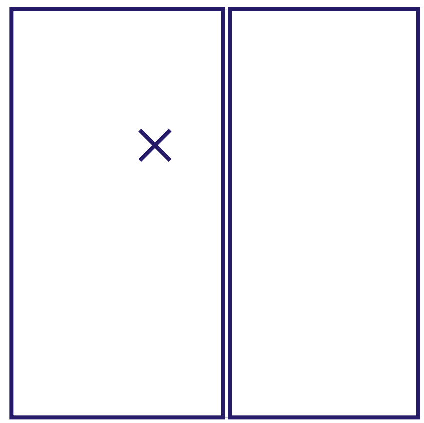
Illustration



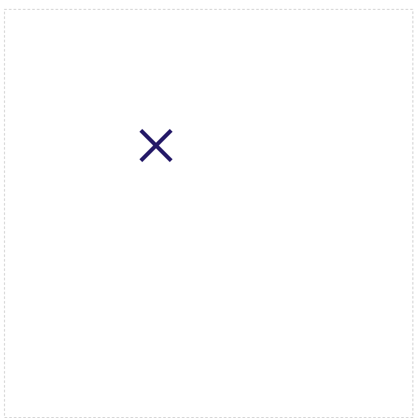
Illustration



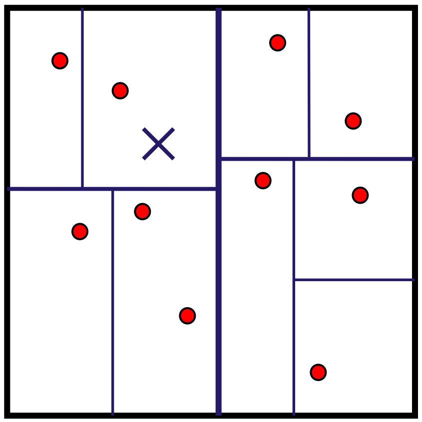
Q_{BB}



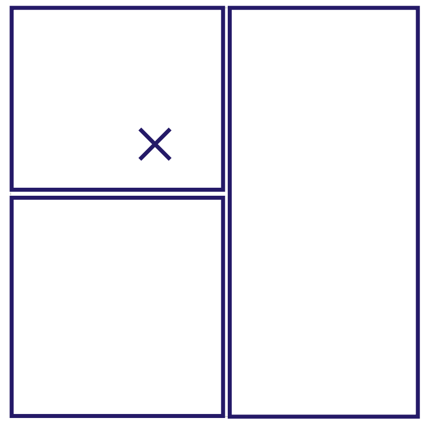
Q_{obj}



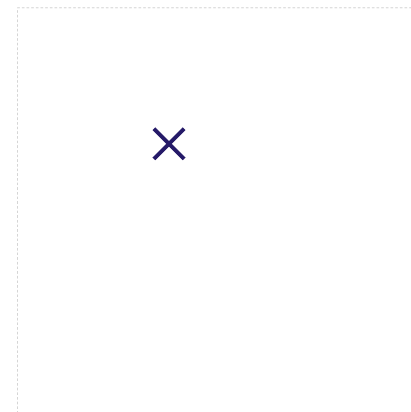
Illustration



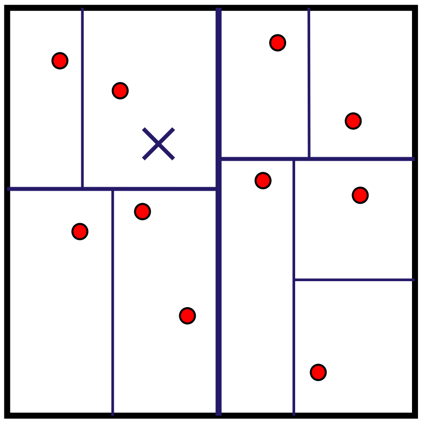
Q_{BB}



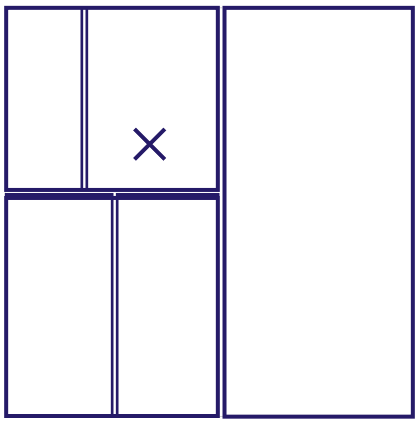
Q_{obj}



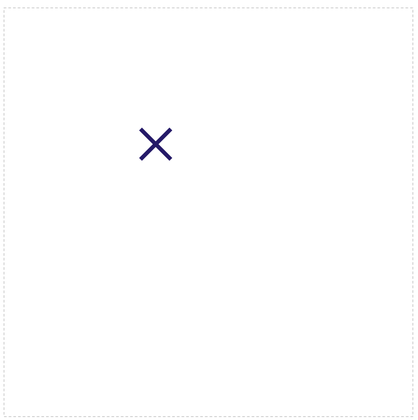
Illustration



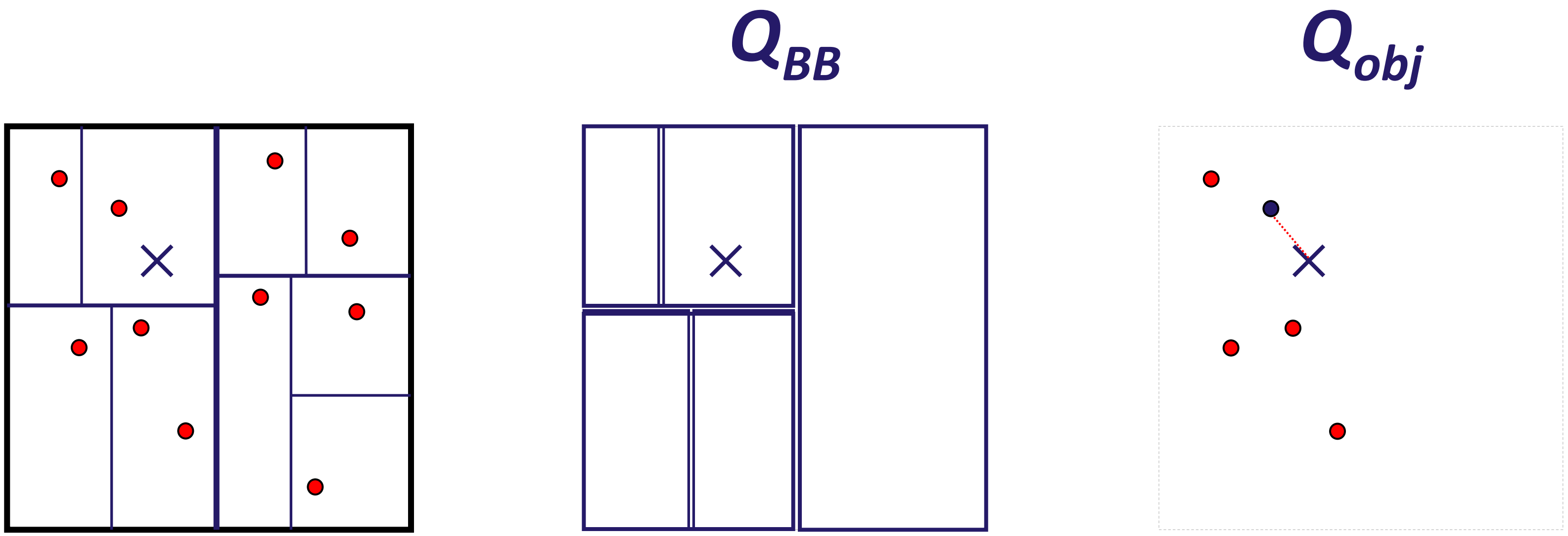
Q_{BB}



Q_{obj}



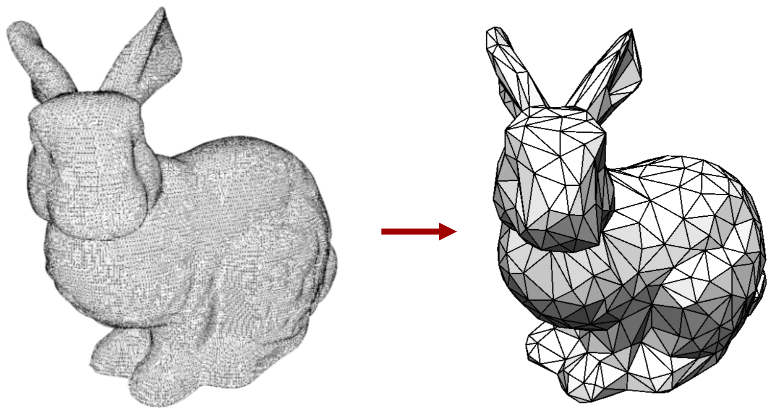
Illustration



Mesh Simplification

Mesh Simplification

Mesh Simplification:



- Triangle meshes are often oversampled
- In particular, meshes from 3D scanners
- We want to decimate the number of triangles such that the shape of the object is roughly maintained
- We want to do this automatically

Variants of the Problem

Problem Variations:

- Mesh simplification
  - Reduce the number of triangles
  - Fixed triangle budget or fixed approximation error
- Multi-resolution models
  - Create a representation that provides many levels of resolution
  - The matching level-of-detail can be extracted at runtime
  - Useful for real-time rendering
    - Choose level of detail for each object in the scene
    - More sophisticated: varying level of detail across one object (the whole scene can be one object)

Curve Simplification

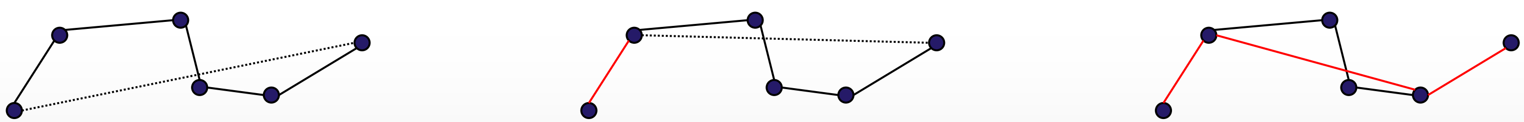
Curve Simplification:

- Compute an approximation of a piecewise linear curve by another piecewise linear curve with fewer segments
- The optimal least-squares solution can be computed in $O(mn^2)$ time using dynamic programming
  - where n = #(*input* line segments)
  - and m = #(*output* line segments)
- Usually, this is still to costly.

Curve Simplification

Curve Simplification:

- Most frequently used heuristic: *Douglas-Peucker Algorithm*.
- Simple Idea:
  - Start with a line connecting the end points
  - Find the input point farthest away from the straight line
  - Insert a new vertex there. We obtain two new segments
  - Apply the algorithm recursively to the parts (a number of times)
- Usually gives (visually) good results



Mesh Simplification

Mesh Simplification:

- We need to find an approximating mesh to a given mesh

Optimal solution?

- It can be shown that finding an L_∞-norm best approximation to a mesh is NP-hard
- For other cases (e.g., least-squares) no efficient optimal techniques are known.

Mesh Simplification

Approximation algorithms:

- Polynomial time approximation algorithms with strict error guarantees are known, but they are too slow for practical applications

Michelangelo's St. Matthew
386,488,573 triangles
[Stanford Digital Michelangelo Project]

Parametric Simplification

If we have a parametric representation

- Spline surface
- Trimmed NURBS
- or the similar

we can just retessellate the original. No need for mesh-based simplification.

In the following: Input is a mesh (no side information)

Mesh Simplification

Three classes of techniques:

- Mesh refinement
  - Start with a simple base mesh, refine to approximate the object
  - “Gift-wrapping”
  - Complicated to implement (need to adjust topology)
- Mesh decimation
  - Start with full mesh
  - Keep on throwing away triangles until precision is met
  - This is the current standard technique
- Other approaches
  - Transform into implicit function and retessellate
  - Vertex clustering on a regular grid (useful for out-of-core impl.)

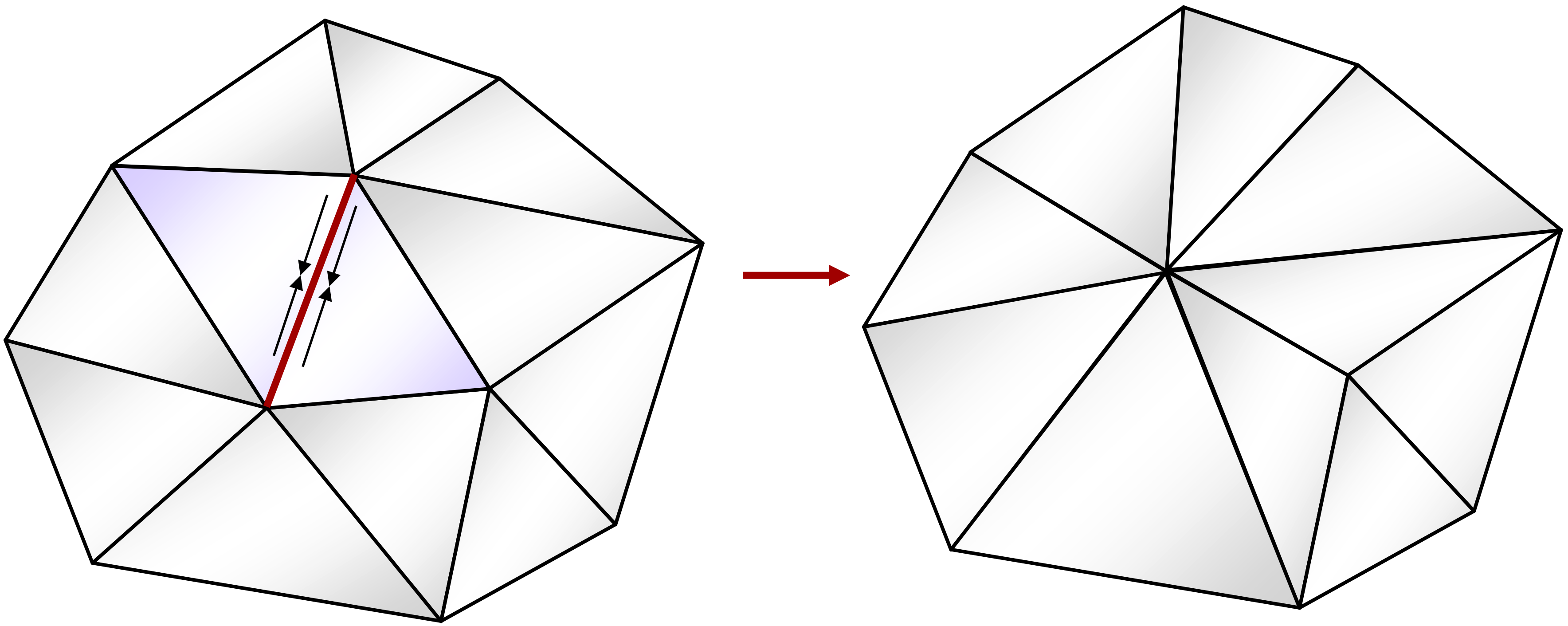
Mesh Decimation

Mesh decimation – basic idea:

- Start with the full mesh
- Then, subsequently remove
  - Triangles (fill hole)
  - Vertices (retriangulate hole)
  - Edges (kills two triangles)
- Edge contraction (“edge collapse”) algorithms are nowadays the most common technique
- Robust and simple to implement

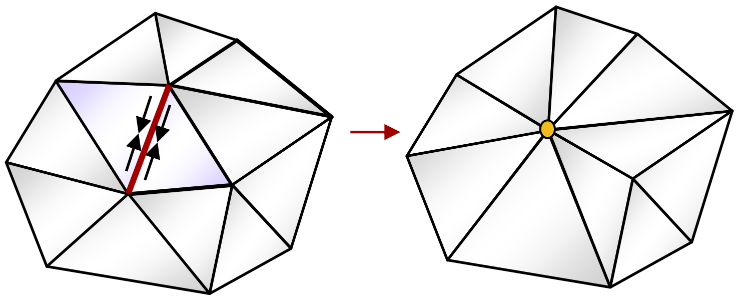
Edge Contraction

Edge contraction:



Edge Contraction

Edge contraction algorithm:



- Questions:
  - Which edges can be collapsed?
  - What error does this cause?
  - Edges collapse into points –
    where should we place the new point?
  - What is the best order for edge collapses?
- Standard algorithm:
  - Greedy algorithm
  - Put edges in priority queue
  - Pick the “cheapest” edge and remove it
  - Recompute costs

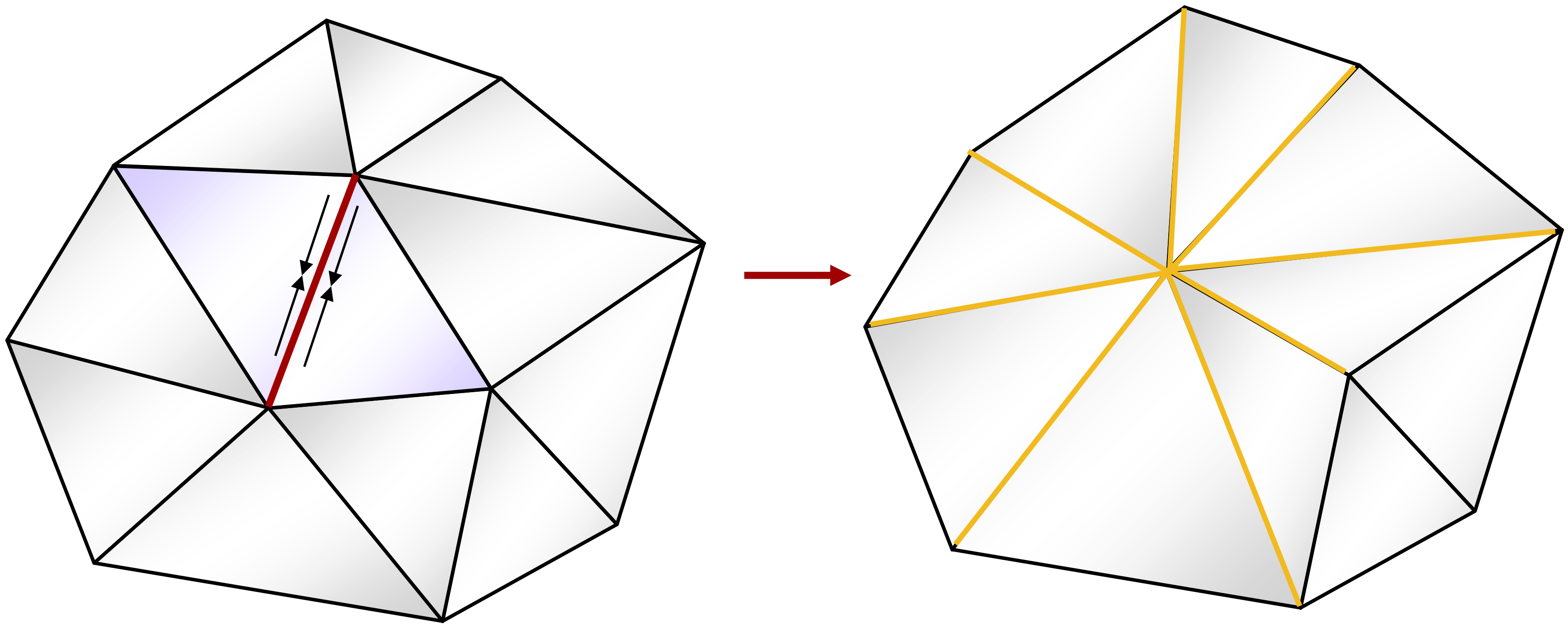
Edge Contraction

Algorithm:

- For each edge in the mesh, compute the costs of collapsing the edge
  - If an edge collapse changes the topology, set costs to $+\infty$
  - Put all (finite cost) edges in priority queue sorted by cost
- **While** queue not empty **and** result not simple enough
  - Remove min-cost edge
  - Collapse the edge
  - Recompute costs of all affected edges (incl. topology check)
  - Update the priority queue accordingly

Edge Contraction

Affected edges:



edge contraction

affected edges

Components

The algorithm needs the following components:

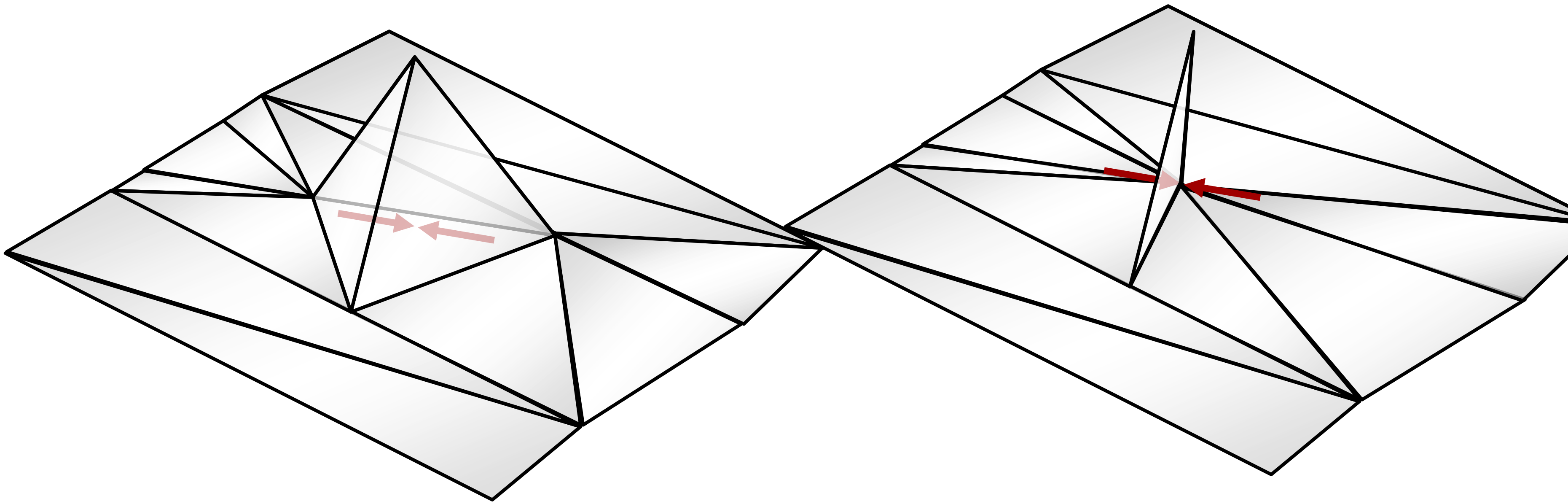
- Topology check (mostly fixed)
- Error metric (lots of choices)
- Placement of new vertices (lots of choices)

Topology Check

We do not want to change the topology of the mesh

- Input is a triangulated two-manifold, probably with boundary
- This means:
  - Every edge is adjacent to *one* or *two* triangles (*boundary* / *interior*)
  - Triangles do not intersect
  - The mesh is conforming – no vertices in the middle of edges (fortunately, edge collapsing cannot change this)

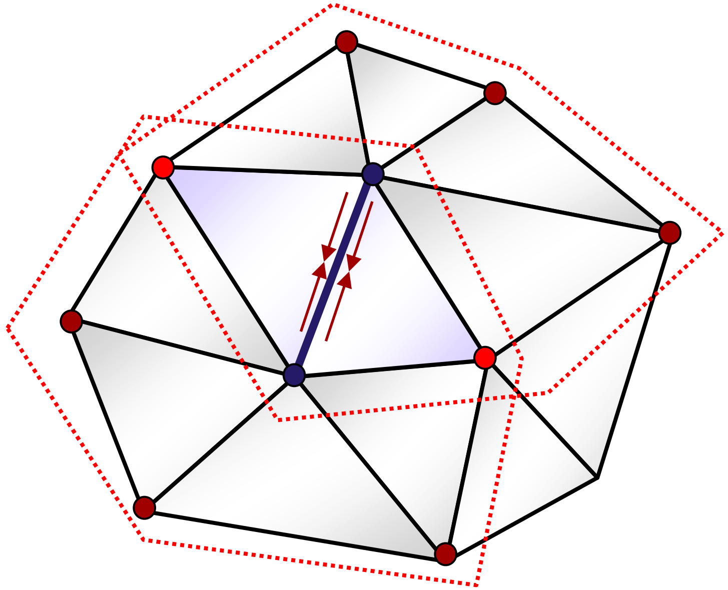
Problem #1: Folds



Problem #1:

- Edge collapses can cause topological folds in meshes
- We need a criterion to prevent this

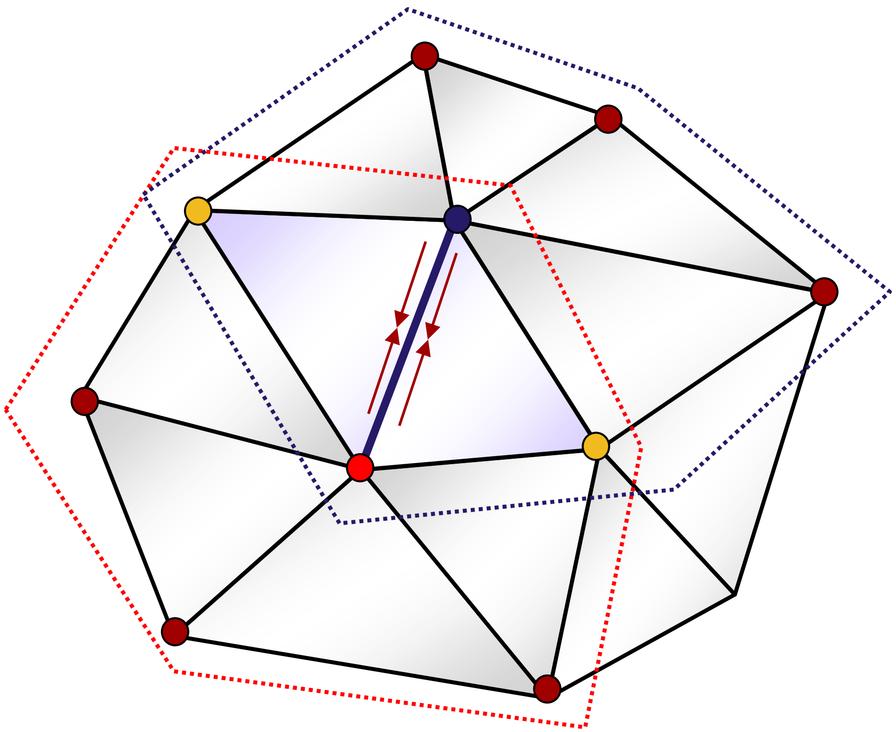
Criterion



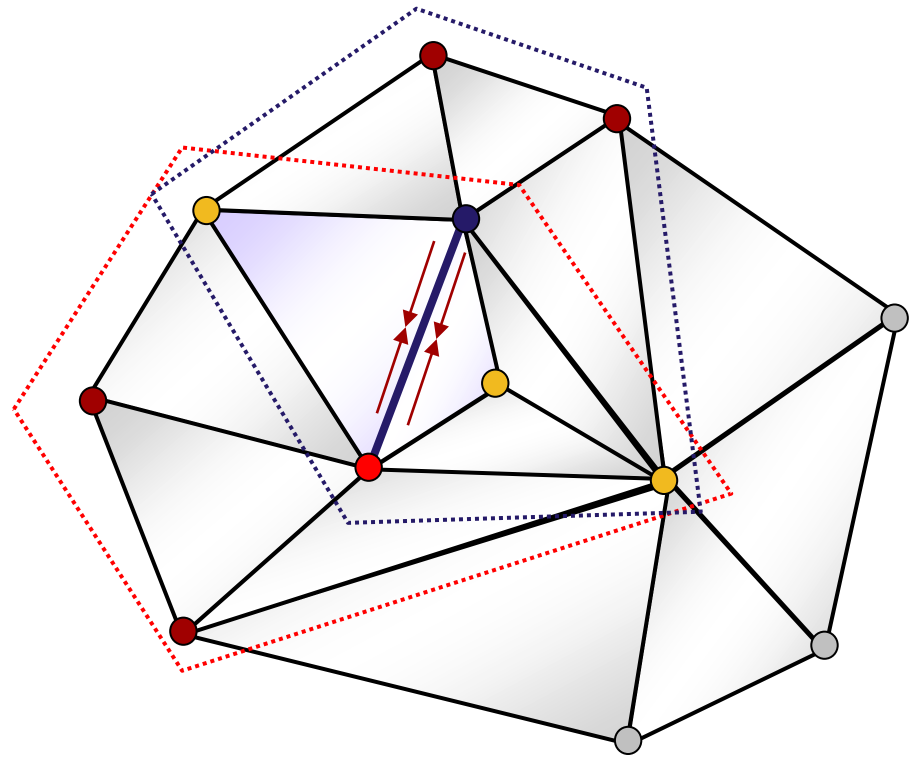
Criterion:

- Consider the two vertices of the edge $\mathbf{v}_1$, $\mathbf{v}_2$
- Let $R^{(1)}(\mathbf{v})$ be the on-ring neighborhood of $\mathbf{v}$, *excluding* $\mathbf{v}_1$, $\mathbf{v}_2$
- If $\#(R^{(1)}(\mathbf{v}_1) \cap R^{(1)}(\mathbf{v}_2)) = 2$, the collapse is permitted
- For boundary points: $\#(R^{(1)}(\mathbf{v}_1) \cap R^{(1)}(\mathbf{v}_2)) = 1$

Illustration



this works



this folds

Intersections

Preventing Intersections

- The previous criterion only guarantees topologically correct meshes
- The embedding into space (read: vertex placement in $\mathbb{R}^3$) can still cause self intersections
- We need to check this separately:
  - Do the newly created triangles intersect with the shape
    - (Hierarchical intersection test with dynamic hierarchy)
  - If so, avoid the collapse operation
- Often, people omit this check (hard to implement, does not happen frequently in practice)

Components

The algorithm needs the following components:

- Topology check (mostly fixed)
- Error metric (lots of choices)
- Placement of new vertices (lots of choices)

Error Metrics

Various potential error metrics:

- S = original, S' = approximation, dist(·,·) = smallest distance
- L_2-error: $\int_S dist(S',x)^2 dx$
- L_1-error: $\int_S |dist(S',x)| dx$
- L_∞-error: $\max_{x \in S} |dist(S',x)|$
- Hausdorff error: $\max\left(\max_{x \in S} |dist(S',x)|, \max_{x \in S'} |dist(S,x)| \right)$

  (two sided maximum distance, symmetric measure)

Complexity Problem

Evaluating the error metric can be expensive:

- Compute the distance between two objects in $\Omega(n+m)$
- Naive computation takes $O(nm)$
- Doing this for each edge collapse is expensive

Solutions:

- Compute distance to previous level of detail only (works well in practice, but no guarantees)
- Use an approximate distance measure.

Quadric Error Metric

Quadric error metric: [Garland and Heckbert 1997]

- Very efficient solution to the error quantification problem
- However, the estimates might be too pessimistic

Idea:

- Measure distance to planes, rather than original triangles
- Collapsed edge results in a point minimizing the error
- The error is represented as a 3D quadric

Quadric Error Metric

Implicit plane equation:

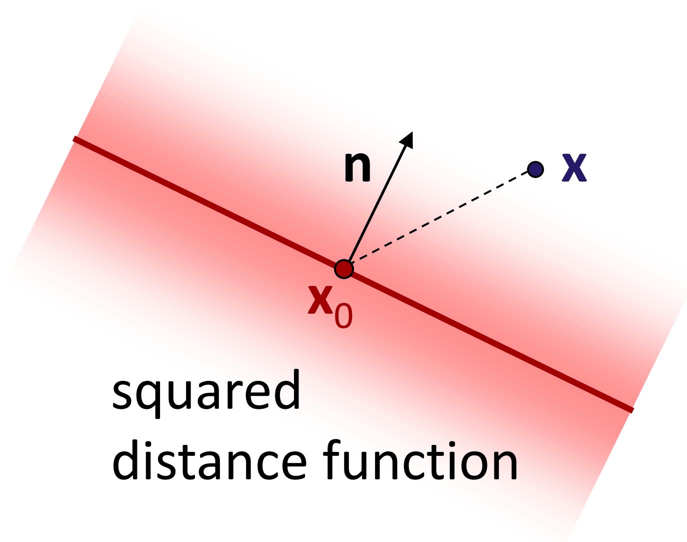
$$\langle \mathbf{n}, \mathbf{x} - \mathbf{x}_0 \rangle = 0$$

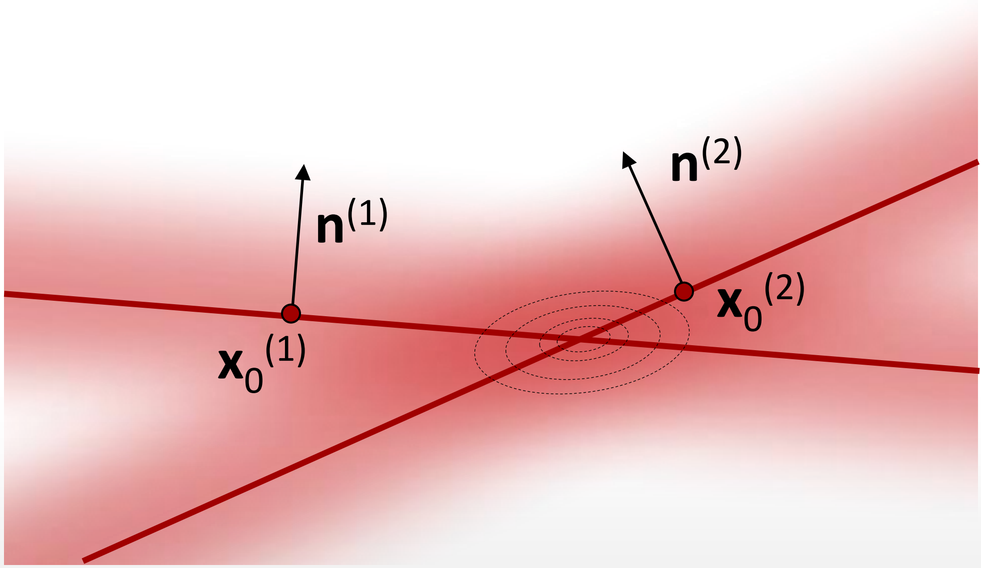
Quadratic error function:

$$\langle \mathbf{n}, \underset{\text{variable}}{\boxed{\mathbf{x}}} - \mathbf{x}_0 \rangle^2$$

Minimum distance to several planes:

$$\sum_{i=1}^{n} \langle \mathbf{n}^{(i)}, \underset{\text{variable}}{\boxed{\mathbf{x}}} - \mathbf{x}_0^{(i)} \rangle^2$$





Quadric Error Metrics

Use in mesh simplification:

- Assign an initial error quadric to each vertex
  - Formed by summing up the plane error functions of the planes of all adjacent triangles
  - Weight components by triangle area
  - Error will be zero for the vertex itself (intersection of all planes)
- For each possible edge contraction:
  - Just add the error quadrics of both vertices involved
  - This means, the new, contracted vertex should approximate the planes of all triangles involved so far as well as possible

Quadric Error Metrics

Use in mesh simplification:

- For each possible edge contraction:
  - Compute the optimum vertex position according to the summed error metric
  - Evaluate the quadric to determine the error
  - This is the candidate move (error, position) that is stored in the edge contraction queue
- When an edge contraction occurs:
  - Use the computed position
  - To recompute neighborhood error quadrics, add the error matrix of the new vertex to each neighboring vertex
  - This gives new edge contraction costs

Extension

Meshes also have attributes, such as:

- Color
- Texture coordinates

This can be handled using quadric error metrics as well:

- Just store additional columns in the x-vectors
- Treat color values (etc.) as additional dimensions of the vertex position, weighted by relative importance to preserve them

How well does this work?

Advantage:

- Very fast: Evaluating the error metric and finding a new vertex position is O(1)

Disadvantage:

- For noisy meshes, the error approximation is bad:

- Possible solutions:
  - Mesh smoothing (normals from larger neighborhoods)
  - Reset quadrics after a few computation steps

Components

The algorithm needs the following components:

- Topology check (mostly fixed)
- Error metric (lots of choices)
- Placement of new vertices (lots of choices)

Conclusion:

- Quadric error metrics are a very popular choice due to their simplicity and performance.
- More accurate alternatives exist (at higher costs).

Multi-Resolution Meshes

Multi-resolution version:

- We want to store multiple levels of detail in one representation
- Simple, but effective approach: Progressive meshes [Hoppe 1996]

Progressive meshes:

- Simplify as strongly as possible (we get a *base mesh*)
- Record all edge contractions in a list

Progressive Meshes

Adjusting the level of detail:

- Start with the base mesh
- Perform *inverse edge contractions*, which are *vertex splits*, to increase the level of detail
- Perform edge contractions to reduce the level of detail
- The index in the list of edge contractions controls the level of detail:
  - Index up: Level of detail increases
  - Index down: Level of detail decreases

Hardware Friendly Implementation

Progressive meshes are expensive:

- Graphics hardware can render billions of triangles
- Performing precomputed edge collapses / vertex splits still takes a lot of computational resources

Hardware Friendly approach:

- Precompute a number of levels of detail
- Just render them as needed
- Use linear interpolation to smoothly blend in the new vertices (avoid popping)