Logic Encodings in LF: A Completeness Criterion

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LF

- LF = A Framework for Defining Logics (Harper, Honsell, Plotkin; 1993)
- Impredicative dependent type theory, related to Martin-Löf type theory
- Curry-Howard equivalent to first-order logic with predicates, implication and universal quantifier
- Types:
  - Application of type-valued constant, e.g., if $\mathit{Matrix}:\mathbb{N} \to \mathbb{N} \to \mathtt{type}$, then $\mathit{Matrix}(5,4):\mathtt{type}$
  - Dependent product, e.g., if $I : \Pi\, x{:}\mathbb{N}.\ \mathit{Matrix}(x,x)$, then $I(3):\mathit{Matrix}(3,3)$ (e.g., identity matrix)

Logic Encodings

- LF very suitable for logic encodings (esspecially natural deduction or sequent calculus)
- Example: Fragment of propositional logic with natural deduction

$$
\begin{aligned}
&\mathit{form} : \mathtt{type}\\
&\wedge : \mathit{form} \to \mathit{form} \to \mathit{form}\\
&\mathit{proof} : \mathit{form} \to \mathtt{type}\\
&\wedge I : \mathit{proof}(F) \to \mathit{proof}(G) \to \mathit{proof}(F \wedge G)\\
&\wedge El : \mathit{proof}(F \wedge G) \to \mathit{proof}(F)\\
&\wedge Er : \mathit{proof}(F \wedge G) \to \mathit{proof}(G)
\end{aligned}
$$

- Structural rules (axiom, weakening, exchange) naturally derivable in the type theory
- Model theory not covered

LF as a Logic

- Proof and model theory for LF developed building on joint work with Steve Awodey
- Permits to
  - encode model theory of logics as well as proof theory
  - formalize encodings as institution translation into LF
  - reason about logic encodings
- Formulas: equalities for all types, first-order connectives, first-order quantifiers for all types, classical negation

Proof theory of the LF meta-logic (examples)

$$\frac{F \vdash_\Sigma F'}{\vdash_\Sigma F \Rightarrow F'} \qquad \frac{\vdash_\Sigma F \Rightarrow F' \quad \vdash_\Sigma F}{\vdash_\Sigma F'}$$

$$\frac{\vdash_\Sigma F \quad \vdash_\Sigma F'}{\vdash_\Sigma F \wedge F'} \qquad \frac{\vdash_\Sigma F \wedge F'}{\vdash_\Sigma F} \qquad \frac{\vdash_\Sigma F \wedge F'}{\vdash_\Sigma F'}$$

$$\frac{x{:}S \vdash_\Sigma F}{\vdash_\Sigma \forall x{:}S.F} \qquad \frac{\vdash_\Sigma \forall x{:}S.F \quad \vdash_\Sigma s : S}{\vdash_\Sigma F[x/s]}$$

$$\frac{\vdash_\Sigma s : S \quad \vdash_\Sigma F[x/s]}{\vdash_\Sigma \exists x{:}S.F} \qquad \frac{x{:}S, F \vdash_\Sigma F'}{\exists x{:}S.F \vdash_\Sigma F'}$$

Example and Completeness Criterion

- Extending LF encodings to cover model theory surprisingly simple
- Example: two axioms needed in the LF meta-logic to encode first-order logic
  - non-empty model universes: $\exists x{:}\mathit{univ}.\mathit{true}$
  - consistency: $\neg\exists x{:}\mathit{proof}(\mathit{false}).\mathit{true}$
- Completeness criterion: All provable existential quantifiers have witnesses

$\boxed{\text{Question: When can this criterion be applied?}}$