Geometric Quantifier Elimination Heuristic for Octagonal Constraints

Deepak Kapur

Department of Computer Science
University of New Mexico
Albuquerque, NM, USA

with Hengjun Zhao (Chinese Academy of Sciences) and
Zhihai Zhang (Peking University)
(work in progress)

Outline

- Quantifier Elimination Approach for Generating (Loop) Invariants.

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- Octagonal Constraints.
- Geometric and Local Quantifier Elimination Heuristic

Invariants: Integer Square Root

Example

```
x := 1, y := 1, z := 0;
while (x <= N) {
  x := x + y + 2;
  y := y + 2;
  z := z + 1
}
return z
```

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    $(I(x, y, z) \wedge x \leq N) \Longrightarrow I(x + y + 2, y + 2, z + 1).$

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  - **VC2:** For every iteration of the loop body:

    $(I(x, y, z) \wedge x \leq N) \Longrightarrow I(x + y + 2, y + 2, z + 1).$
- Using quantifier elimination, find constraints on parameters $A, B, C, D, E, F, G, H, J, K$ which ensure that the verification conditions are valid for all possible program variables.

- Constraints on parameters are:

$$C = -F, \quad J = -2B - F + 2K, \quad G = -4B - F, \quad H = 3B + F - K.$$

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from which $x = (z + 1)^2$ follows.

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- Generate verification conditions for every path in the program (a path from an assertion to another assertion including itself).
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- Find a formula expressed in terms of parameters eliminating all program variables (using quantifier elimination).

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Soundness and Completeness

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  - If no parameter values can be found, then invariants of hypothesized forms may not exist. Invariants can be guaranteed not to exist if no approximations are made, while generating verification conditions.
- If all assignments making the formula true can be finitely described, invariants generated may be the strongest of the hypothesized form. Invariants generated are guaranteed to be the strongest if no approximations are made, while generating verification conditions.

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- Reduction Approach to Decision Procedures for Theories over Abstract Data Structures, including Finite Lists, Finite Sets, Finite Arrays, Finite Multisets (Kapur and Zarba, 2005).

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- Extensively investigated in many areas including program analysis, program synthesis, termination of programs, as well as hybrid system analysis, particularly safety check and controller synthesis.
- Closely related to choosing an abstract domain in the abstract interpretation approach.

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  - Output is huge and difficult to decipher.
  - In practice, they often do not work (i.e., run out of memory or hang).
- Linear constraint solving on rationals and reals (polyhedral domain), while of polynomial complexity, has been found in practice to be inefficient and slow, especially when used repeatedly as in abstract interpretation approach [Miné]

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- Miné gave well-designed algorithms based on Difference Bound Matrices (DBMs) and graph representation for performing various operations needed for program analysis using the abstract interpretation approach.
- Miné's algorithms are of $O(n^3)$ (sometimes, $O(n^4)$), where n is the number of variables.

Octagonal Constraints and Quantifier Elimination

- Octagonal constraints have a fixed shape. Given n variables, the most general formula (after simplification) is of the following form
$\bigwedge_{i,j}(\quad l_{i,j} : \quad a_{i,j} \leq x_i - x_j \leq b_{i,j}, \quad c_{i,j} \leq x_i + x_j \leq d_{i,j}, \quad e_i \leq x_i \leq f_i \quad g_j \leq x_j \leq h_j)$

  for every pair of variables x_i, x_j, where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, e_i, f_i, g_j, h_j$ are parameters.

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  for every pair of variables x_i, x_j, where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, e_i, f_i, g_j, h_j$ are parameters.
- For a finite program path consisting of a sequence of assignment statements interspersed with tests, its behavior is approximated so that the post condition is also of the above form.
- A verification condition is expressed using atomic formulas that are all octagonal constraints.

$$\bigwedge_{i,j}((I_{i,j} \wedge \alpha(x_i, x_j)) \Rightarrow I'_{i,j}),$$

  along with additional parameter-free constraints $\alpha(x_i, x_j)$, of the same form in which lower and upper bounds are constants.

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- Quantifier elimination heuristics can be developed using which it is possible to generate constraints on lower and upper bounds by table look ups in $O(n^2)$ steps, where n is the number of program variables.

Geometric QE Heuristic

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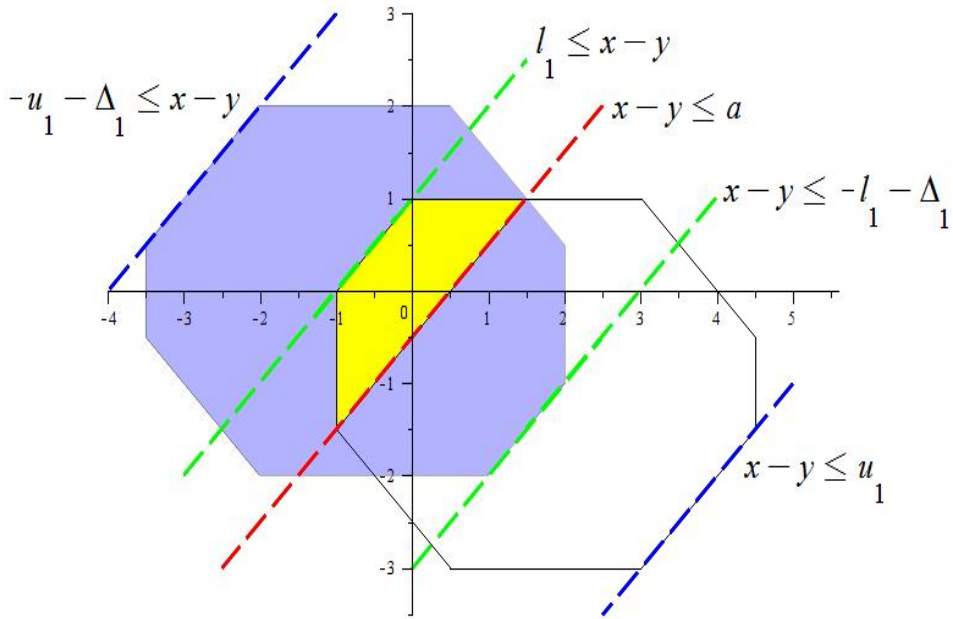
- Identify conditions under which the transformed octagon includes the portion of the original octagon satisfying tests along a program path.
- For each assignment case, a table is built showing the effect on the parameter values by determining the effect on every type of constraints.

Table 1: Assignments with signs of variables reversed

$x_i := -x_i + A, \quad x_j := -x_j + B, \quad \Delta_1 = B - A, \quad \Delta_2 = -A - B, \quad \Delta_3 = -A, \quad \Delta_4 = -B.$



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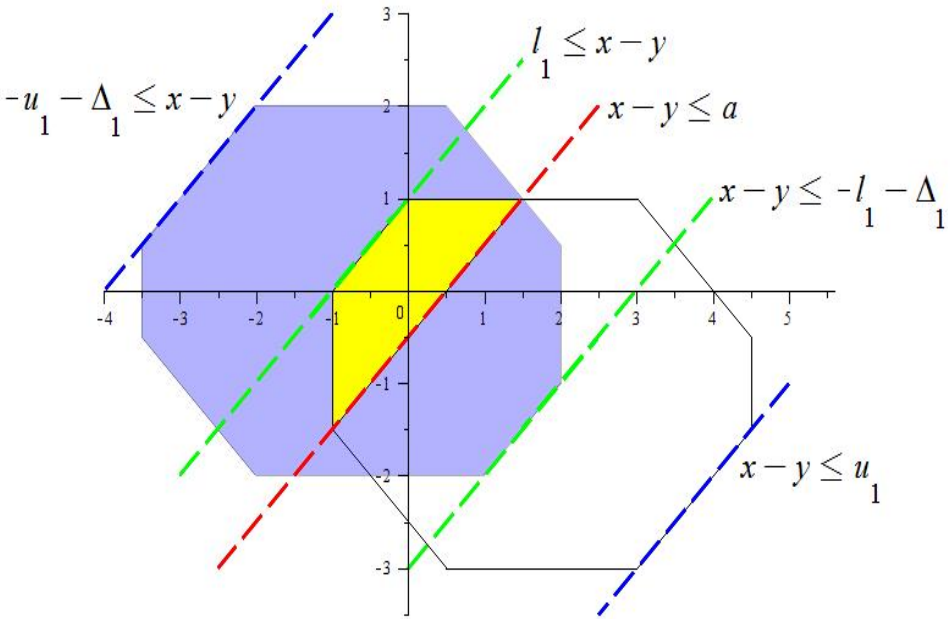

	present	absent
$x_i - x_j \le a$	$a \le u_1$ $a \le -l_1 - \Delta_1$	$u_1 \le -l_1 - \Delta_1$
$x_i - x_j \ge b$	$l_1 \le b$ $-u_1 - \Delta_1 \le b$	$-u_1 - \Delta_1 \le l_1$
$x_i + x_j \le c$	$c \le u_2$ $c \le -l_2 - \Delta_2$	$u_2 \le -l_2 - \Delta_2$
$x_i + x_j \ge d$	$l_2 \le d$ $-u_2 - \Delta_2 \le d$	$-u_2 - \Delta_2 \le l_2$
$x_i \le e$	$e \le u_3$ $e \le -l_3 - \Delta_3$	$u_3 \le -l_3 - \Delta_3$
$x_i \ge f$	$l_3 \le f$ $-u_3 - \Delta_3 \le f$	$-u_3 - \Delta_3 \le l_3$
$x_j \le g$	$g \le u_4$ $g \le -l_4 - \Delta_4$	$u_4 \le -l_4 - \Delta_4$
$x_j \ge h$	$l_4 \le h$ $-u_4 - \Delta_4 \le h$	$-u_4 - \Delta_4 \le l_4$

Table 2: No changing signs of variables

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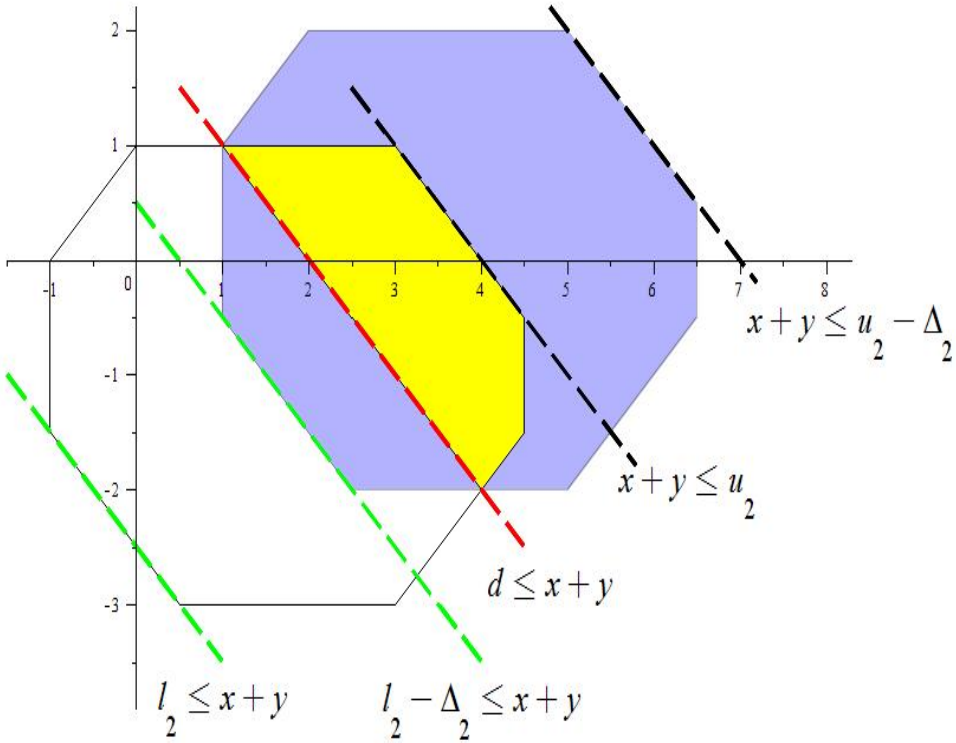

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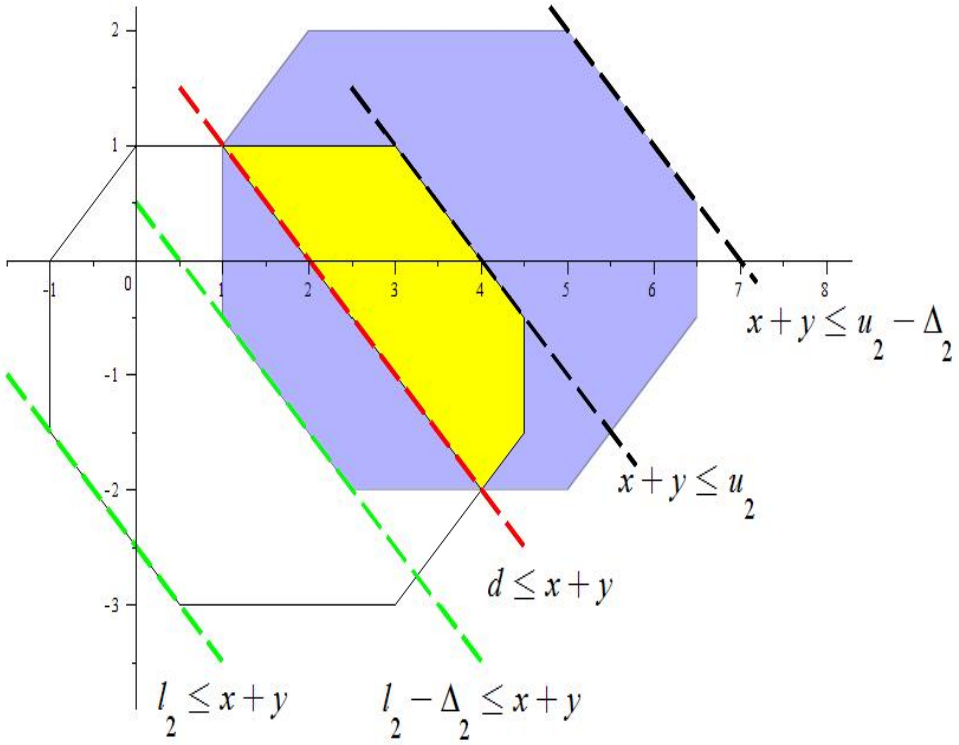

	present	absent
$x_i - x_j \le a$ $\Delta_1 > 0$	$u_1 \ge a + \Delta_1$	$u_1 = +\infty$
$x_i - x_j \ge b$ $\Delta_1 < 0$	$l_1 \le b + \Delta_1$	$l_1 = -\infty$
$x_i + x_j \le c$ $\Delta_2 > 0$	$u_2 \ge c + \Delta_2$	$u_2 = +\infty$
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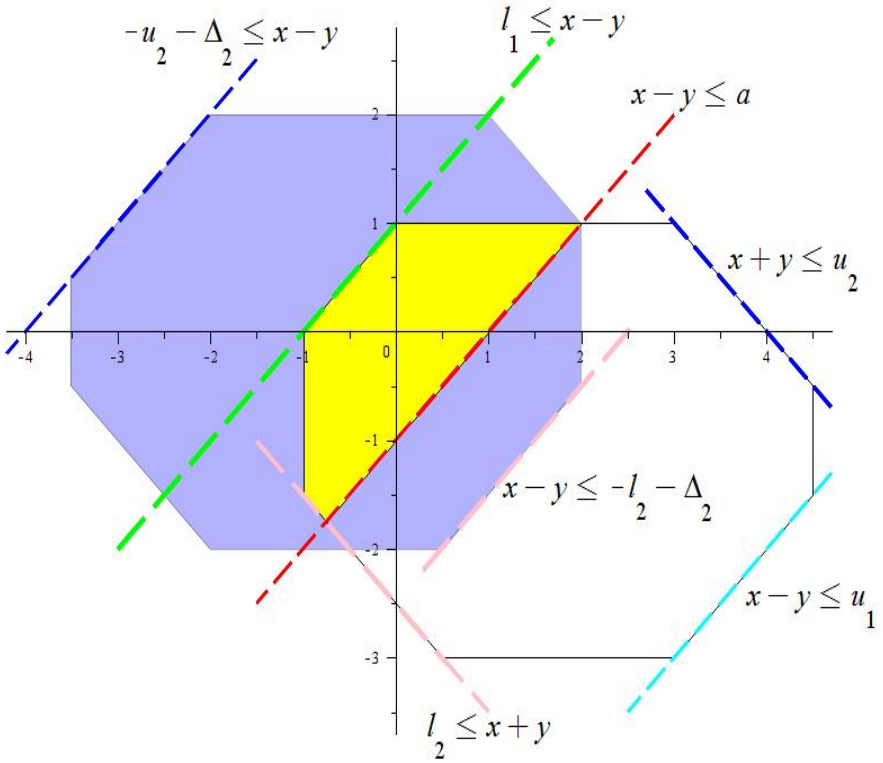
$x_i \ge f$ $\Delta_3 < 0$	$l_3 \le f + \Delta_3$	$l_3 = -\infty$
$x_j \le g$ $\Delta_4 > 0$	$u_4 \ge g + \Delta_4$	$u_4 = +\infty$
$x_j \ge h$ $\Delta_4 < 0$	$l_4 \le h + \Delta_4$	$l_4 = -\infty$

Table 3: Sign of exactly one variable is changed

$x_i := -x_i + A, \quad x_j := x_j + B, \quad \Delta_1 = B - A, \quad \Delta_2 = -A - B, \quad \Delta_3 = -A, \quad \Delta_4 = B.$



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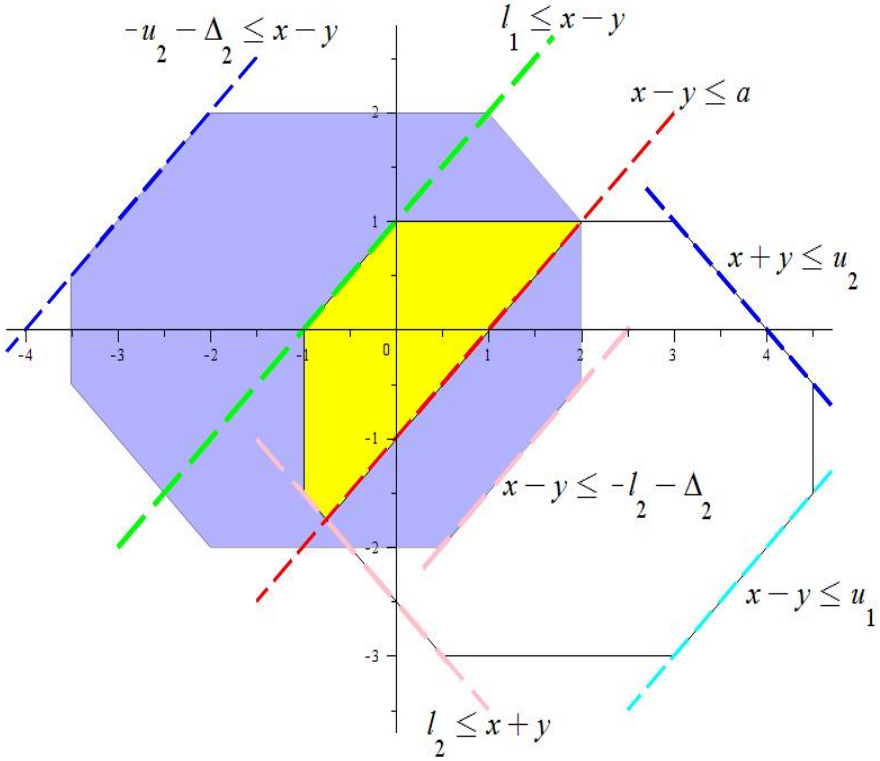

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$x_j \le g$ $\Delta_4 > 0$	$u_4 \ge g + \Delta_4$	$u_4 = +\infty$
$x_j \ge h$ $\Delta_4 < 0$	$l_4 \le h + \Delta_4$	$l_4 = -\infty$

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- The above can be computed from the frame representation of the octagons.

A Simple Example

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```
 x := 4; y := 6;
while (x + y >= 0) do
 if (y >= 6) then { x := -x;  y := y - 1 }
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VC0: $I(4, 6)$

VC1: $(I(x, y) \wedge (x + y) \geq 0 \wedge y \geq 6) \Longrightarrow I(-x, y - 1)$.

VC2: $(I(x, y) \wedge (x + y) \geq 0 \wedge y < 6) \Longrightarrow I(x - 1, -y)$.

Generating Constraints on Parameters

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- The corresponding invariant is:

  $$-10 \leq x - y \leq 9 \wedge -11 \leq x + y \leq 10 \wedge -6 \leq x \leq 6 \wedge -5 \leq y \leq 6.$$

Local QE Heuristics: Should we propagate?

- Local propagation (i.e., propagate bounds only for the pair of variables appearing in the constraints) on tests to bring them in a canonical form can sometimes improve the bounds. But that is not always clear. The complexity still remains $O(n^2)$.

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- There are examples for which both local and global propagation (closure) lead to worse results.

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- It has been found in practice that QE based methods generate stronger inductive invariants than methods based on abstract interpretation for polyhedral domain.
- An open question: the strength of invariants generated by the proposed incomplete heuristics.

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- Since general (complete) QE methods are very expensive and their outputs are hard to decipher, it is better to consider special cases, sacrificing completeness as well as generality.
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- Many bells and whistles are needed, just like in the abstract interpretation approach.
- An implementation will determine whether the approach is effective.