Labelings for Decreasing Diagrams

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Question

Wikimedia

Overview

- Preliminaries
- Decreasing Diagrams
- Labelings for Decreasing Diagrams
- Experiments
- Conclusion

Preliminaries

Definition (ARS)

$\mathcal{A} = (A, \to)$

Definition (confluence)

${}^*\!\leftarrow \cdot \to^* \;\subseteq\; \to^* \cdot {}^*\!\leftarrow$

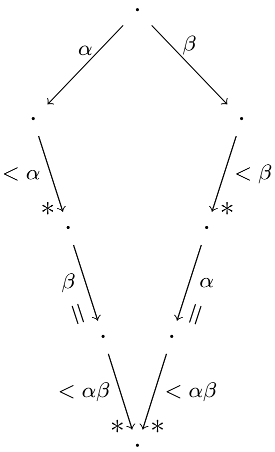
Definition (local confluence)

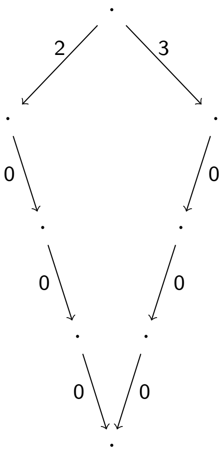
$\leftarrow \cdot \to \;\subseteq\; \to^* \cdot {}^*\!\leftarrow$

Theorem (van Oostrom, 1994)

every locally decreasing ARS is confluent

Decreasing Diagrams – Examples





Term Rewriting

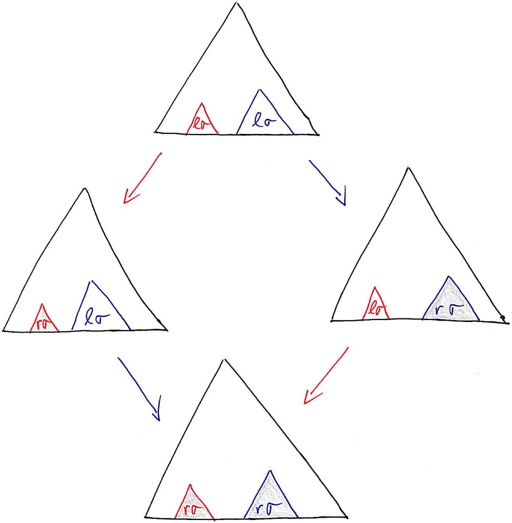
$t \leftarrow s \rightarrow u$

Three possibilities (modulo symmetry)

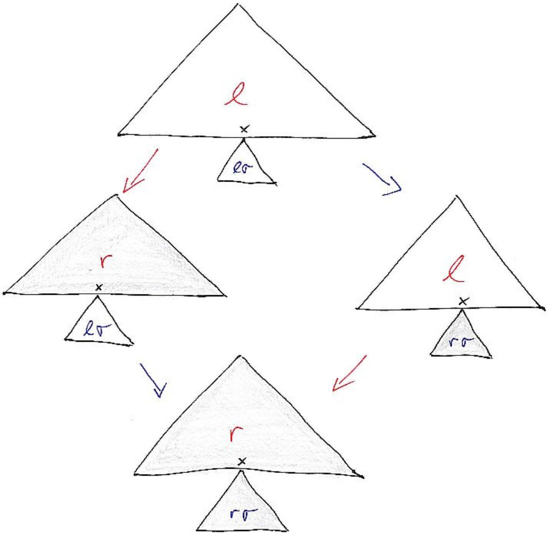
$t \;{}_{p,l \to r}\!\!\leftarrow s[l\sigma]_p = s = s[l\sigma]_q = l \rightarrow_{q,l \to r} u$

- $p \parallel q$ (parallel overlap)
- $p < q$ and $q \leqslant \mathcal{P}\mathrm{os}_{\mathcal{V}}(s[l]_p)$ (variable overlap)
- $p \leqslant q$ and $q \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(s[l]_p)$ (critical overlap)

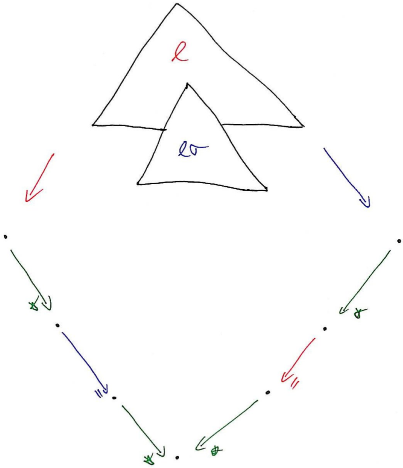
parallel



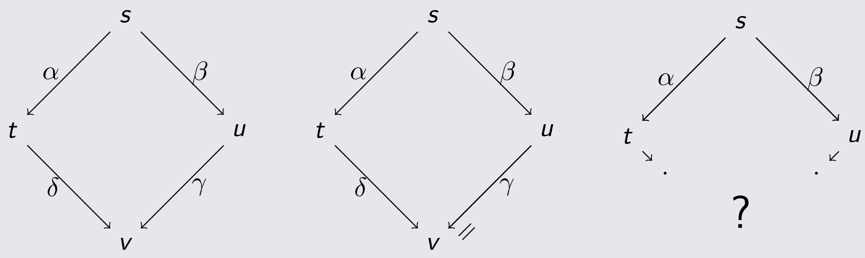
variable (linear)



critical



linear TRSs (3 kinds of local peaks)



(a) parallel (b) variable linear (c) critical

Definition (labeling)

$\Gamma = s \to_{p,l\to r} t \qquad \Delta = u \to_{q,l'\to r'} v$

$(\ell, \geqslant, >)$ labeling if $\geqslant \cdot > \cdot \geqslant \subseteq >$

- $\ell(\Gamma) \geqslant \ell(\Delta) \longrightarrow \ell(C[\Gamma\sigma]) \geqslant \ell(C[\Delta\sigma])$
- $\ell(\Gamma) > \ell(\Delta) \longrightarrow \ell(C[\Gamma\sigma]) > \ell(C[\Delta\sigma])$

L-labeling

Definition (L-labeling)

labeling ℓ is L-labeling if $\alpha \geqslant \gamma$, $\beta \geqslant \delta$ (parallel, variable linear)

Example

$(\ell_{\mathsf{rl}}, \geqslant_{\mathbb{N}}, >_{\mathbb{N}})$ is L-labeling ($\ell_{\mathsf{rl}} : \mathcal{R} \to \mathbb{N}$)
$(\ell_{\mathsf{sn}}, \to_{\mathcal{R}}^{*}, \to_{\mathcal{S}/\mathcal{R}}^{+})$ is L-labeling ($\ell_{\mathsf{sn}}(s \to t) = s$)
if $\to_{\mathcal{S}} \subseteq \to_{\mathcal{R}}$ and $\mathcal{S}/\mathcal{R}$ is terminating

Lemma

ℓ_1, ℓ_2 *L-labelings* $\longrightarrow$ $\ell_1 \times \ell_2$ *L-labeling*

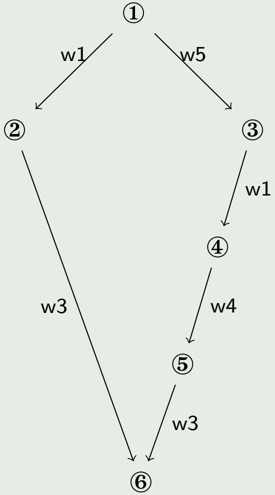
Theorem

$\mathcal{R}$ *linear, critical diagrams decreasing for L-labeling* $\longrightarrow$ $\mathcal{R}$ *confluent*

Example (van Oostrom, 2008)

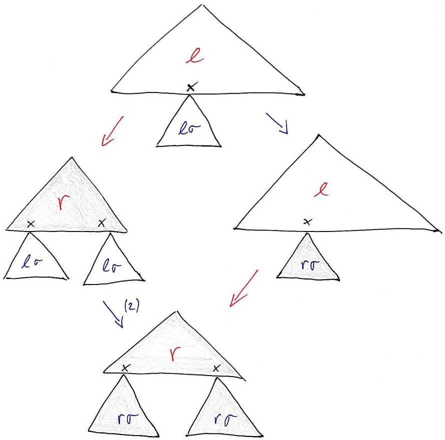
$$
\begin{aligned}
1 :&\quad \text{nats} \to 0 : \text{inc}(\text{nats})\\
2 :&\quad \text{hd}(x : y) \to x\\
3 :&\quad \text{tl}(x : y) \to y\\
4 :&\quad \text{inc}(x : y) \to \text{s}(x) : \text{inc}(y)\\
5 :&\quad \text{inc}(\text{tl}(\text{nats})) \to \text{tl}(\text{inc}(\text{nats}))
\end{aligned}
$$

① inc(tl(nats))
② inc(tl(0 : inc(nats)))
③ tl(inc(nats))
④ tl(inc(0 : inc(nats)))
⑤ tl(s(0) : inc(inc(nats)))
⑥ inc(inc(nats))



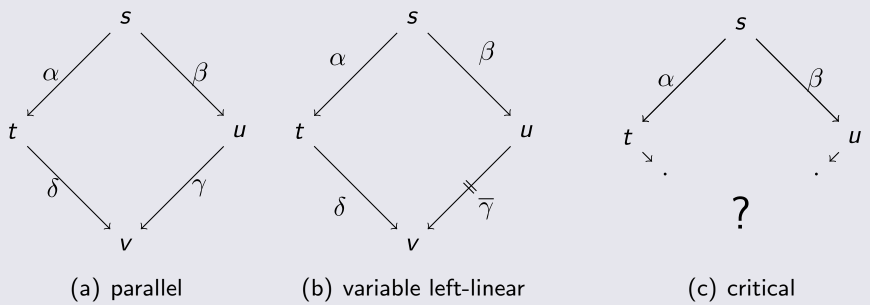
$w1 >_{\mathbb{N}} w3, w4$ shows confluence

variable (left-linear)



LL-labeling

left-linear TRSs (3 kinds of local peaks)



Definition (*LL*-labeling)

L-labeling ℓ is *LL*-labeling if $\alpha > \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

Example

ℓ_{rl} is not LL-labeling
ℓ_{sn} is LL-labeling if $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ terminating

Definition (LL-labeling)

L-labeling ℓ is weak LL-labeling if $\alpha \geqslant \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

Example

ℓ_{rl} is weak LL-labeling
every LL-labeling is weak LL-labeling

Lemma

LL-labeling ℓ_1, *weak LL-labeling* $\ell_2 \longrightarrow \ell_1 \times \ell_2$, $\ell_2 \times \ell_1$ *LL-labelings*

Theorem

$\mathcal{R}$ *left-linear, critical diagrams decreasing for LL-labeling* $\longrightarrow \mathcal{R}$ *confluent*

First result

Corollary

$\mathcal{R}$ left-linear, $\mathcal{R}_d/\mathcal{R}_{nd}$ terminating, critical diagrams decreasing for weak LL-labeling $\longrightarrow$ $\mathcal{R}$ confluent

Example (OO03 cont'd)

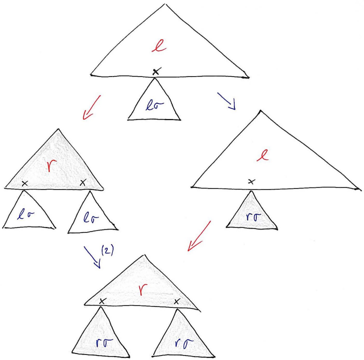
$$
\begin{array}{llll}
1: & x + (y + z) \to (x + y) + z & 2: & (x + y) + z \to x + (y + z) \\
3: & \mathsf{s}(x) + y \to x + \mathsf{s}(y) & 4: & x + \mathsf{s}(y) \to \mathsf{s}(x) + y \\
5: & x \times \mathsf{s}(y) \to x + (x \times y) & 6: & \mathsf{s}(x) \times y \to (x \times y) + y \\
7: & x + y \to y + x & 8: & x \times y \to y \times x \\
9: & \mathsf{sq}(x) \to x \times x & 10: & \mathsf{sq}(\mathsf{s}(x)) \to (x \times x) + \mathsf{s}(x + x)
\end{array}
$$

$\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ terminating

$+_{\mathbb{N}}(x, y) = x + y \quad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \quad \times_{\mathbb{N}}(x, y) = x^2 + xy + y^2 \quad \mathsf{sq}_{\mathbb{N}}(x) = 3x^2 + 1$

weak *LL*-labeling $\ell_{\mathsf{rl}}(8) = \ell_{\mathsf{rl}}(9) = 2$, $\ell_{\mathsf{rl}}(6) = \ell_{\mathsf{rl}}(10) = 1$, $\ell_{\mathsf{rl}}(\cdot) = 0$

Towards a second result



Example

$1 : f(g(x, a)) \to g(f(x), f(x)) \quad 2 : a \to b \quad 3 : b \to a \quad 4 : h(x) \to h(x)$

$\mathcal{R}^\star_> = \{f_1(g_1(x)) \to g_1(f_1(x)), f_1(g_1(x)) \to g_2(f_1(x))\}$
$\mathcal{R}^\star_= = \{h_1(x) \to h_1(x)\}$

Definition

$\ell_\star(s \to_{p,l\to r} t) = \star(s,p)$

Lemma

$\mathcal{R}^\star_> / \mathcal{R}^\star_=$ *terminating* $\longrightarrow$ $(\ell_\star, \to^*_{\mathcal{R}^\star_=}, \to^+_{\mathcal{R}^\star_>/\mathcal{R}^\star_=})$ *LL-labeling*

Corollary

$\mathcal{R}$ *left-linear,* $\mathcal{R}^\star_> / \mathcal{R}^\star_=$ *terminating,* ℓ *weak LL-labeling, critical peaks decreasing for* $\ell_\star \times \ell$ $\longrightarrow$ $\mathcal{R}$ *confluent*

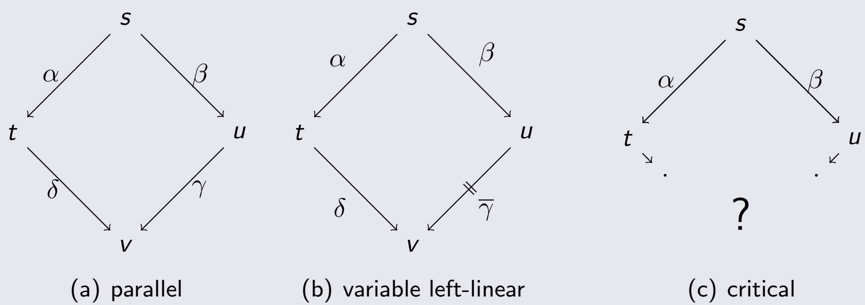
Example (OO03)

$$\begin{array}{llll}
1: & x + (y + z) \to (x + y) + z & 2: & (x + y) + z \to x + (y + z) \\
3: & \mathsf{s}(x) + y \to x + \mathsf{s}(y) & 4: & x + \mathsf{s}(y) \to \mathsf{s}(x) + y \\
5: & x \times \mathsf{s}(y) \to x + (x \times y) & & \dots
\end{array}$$

$\mathcal{R}^\star_>$ contains nonterminating $\times_1(x) \to +_2(\times_1(x))$

LL-labeling

left-linear TRSs (3 kinds of local peaks)



Definition (*LL*-labeling)

L-labeling ℓ is *LL*-labeling if $\alpha > \gamma_i$, $\beta \geqslant \delta$ (variable left-linear)

$\alpha \geqslant \gamma_1$, $\alpha > \gamma_i$ $(i > 1)$

Example (OO03)

$\mathcal{R}^\star_>$

$$\begin{aligned}
\mathsf{sq}_1(\mathsf{s}_1(x)) &\to +_1(\times_1(x))\\
\mathsf{sq}_1(\mathsf{s}_1(x)) &\to +_1(\times_2(x))\\
\mathsf{sq}_1(\mathsf{s}_1(x)) &\to +_2(\mathsf{s}_1(+_1(x)))\\
\mathsf{sq}_1(\mathsf{s}_1(x)) &\to +_2(\mathsf{s}_1(+_2(x)))\\
\mathsf{sq}_1(x) &\to \times_1(x)\\
\mathsf{sq}_1(x) &\to \times_2(x)\\
\times_2(y) &\to +_1(\times_2(y))\\
\times_2(y) &\to +_2(y)\\
\times_1(x) &\to +_1(x)\\
\times_1(x) &\to +_2(\times_1(x))
\end{aligned}$$

$\mathcal{R}^\star_=$

$$\begin{aligned}
\times_1(x) &\to \times_2(x)\\
\times_2(y) &\to \times_1(y)\\
+_1(x) &\to +_2(x)\\
+_2(y) &\to +_1(y)\\
+_1(x) &\to +_1(\mathsf{s}_1(x))\\
+_1(x) &\to +_1(+_1(x))\\
+_2(z) &\to +_2(+_2(z))\\
+_2(y) &\to +_2(\mathsf{s}_1(y))\\
\times_1(x) &\to +_2(\times_1(x))
\end{aligned}$$

$$\begin{aligned}
\times_1(\mathsf{s}_1(x)) &\to +_1(\times_1(x))\\
\times_2(\mathsf{s}_1(y)) &\to +_2(\times_2(y))\\
+_1(\mathsf{s}_1(x)) &\to +_1(x)\\
+_1(+_1(x)) &\to +_1(x)\\
+_1(+_2(y)) &\to +_2(+_1(y))\\
+_2(+_2(z)) &\to +_2(z)\\
+_2(\mathsf{s}_1(y)) &\to +_2(y)\\
+_2(+_1(y)) &\to +_1(+_2(y))\\
\times_2(y) &\to +_1(\times_2(y))
\end{aligned}$$

$\mathsf{sq}_{1\mathbb{N}}(x) = x + 2 \quad \times_{1\mathbb{N}}(x) = \times_{2\mathbb{N}}(x) = x + 1 \quad +_{1\mathbb{N}}(x) = +_{2\mathbb{N}}(x) = \mathsf{s}_1(x) = x$

Question

Google Maps

C S I

Experiments

53 left-linear TRSs (30 years of confluence literature)

method	pre	CR(ℓ_{rl})	CR(ℓ_{sn})	CR
rule labeling	40	35	–	35
SN($\mathcal{R}_d/\mathcal{R}_{nd}$)	45	40	37	42
SN($\mathcal{R}^{\star}_{>}/\mathcal{R}^{\star}_{=}$)	45	42	34	42
SN($\mathcal{R}^{\star\star}_{>}/\mathcal{R}^{\star\star}_{=}$)	48	45	36	45
ACP				48
CSI				49

Conclusion

Summary

- decreasing diagrams
- (linear) ℓ_{rl}, ℓ_{sn}, $\ell_\star$
- (left-linear) $\mathsf{SN}(\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}})$, $\mathsf{SN}(\mathcal{R}^\star_>/\mathcal{R}^\star_=)$, $\mathsf{SN}(\mathcal{R}^{\star\star}_>/\mathcal{R}^{\star\star}_=)$

Future Work

- study parallel reduction