Automatic Analysis of Hybrid Systems

A Constraint-Solving Perspective

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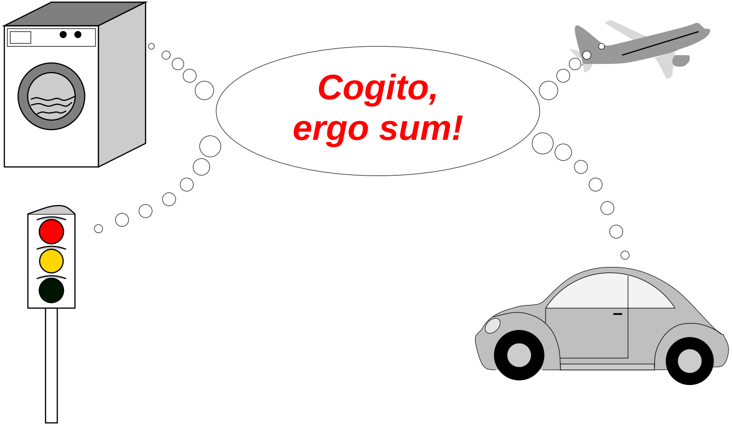
SFB/TR 14 AVACS

Agenda

1. Embedded systems
   - the term
   - automata-based verification
2. Hybrid discrete-continuous systems
   - the term
   - need for a dedicated theory
3. Automatic methods for hybrid-system analysis
   - Reachability analysis by iterating the transition relation
     - Decidable cases (very confined...)
     - Decidable wrt. depth-bounded reachability
     - Undecidable even in the bounded (the truly interesting stuff...)
   - Some other verification schemes

Embedded computer systems



Estimates for number of embedded systems in use exceed 10^{10}.

[Rammig 2000, Motorola 2001]

Application domains

- **Consumer & household products:**
  CD players, TV sets, handheld games, electronic pets, cameras, alarm clocks, remote controls, dishwashers, microwave ovens, ...
- **Office, telecommunications, etc.:**
  Printers, network controllers, mobile phones, keyboards, CRTs and flatscreens, ...
- **Environmental control:**
  λ control, programmable heating systems, exhaust control, ...
- **Traffic systems and traffic management:**
  Cars (body, powertrain, suspension, brakes), signalling devices, balises, interlocks, autopilots, traffic information, ...
- **Medicine:**
  Measurement devices (thermometers, RR's, X-ray, sonographic imaging, EEG, ECG ...), treatment devices (perfusors, respirators, microwave radiation treatment, ...)
- **Supplies:**
  Power plants, distribution networks, ...

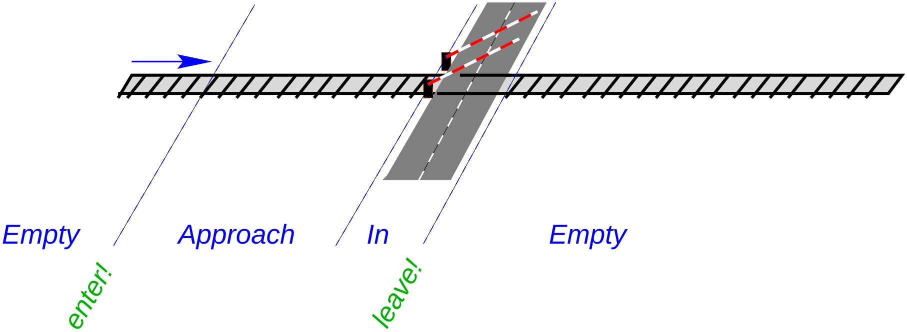
Formal Methods

- Formal methods are mathematically-based techniques for the specification, development and verification of software and hardware systems. [R.W. Butler, 2001]
- Motivated by the expectation that appropriate mathematical analyses can contribute to the reliability and robustness of a design. [M. Holloway, 1997]
- Alternative to less exhaustive analyses:

[cartoon]

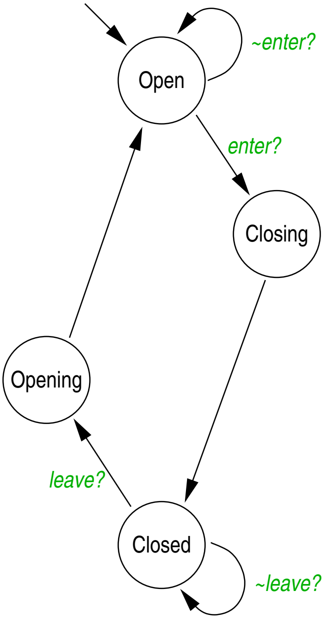
Automatic Analysis of Embedded Systems

A first idea:
Automata-theoretic verification



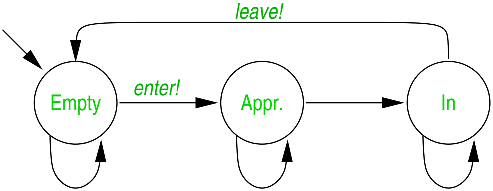
Safety requirement: Gate has to be closed whenever a train is in "In".

The gate model

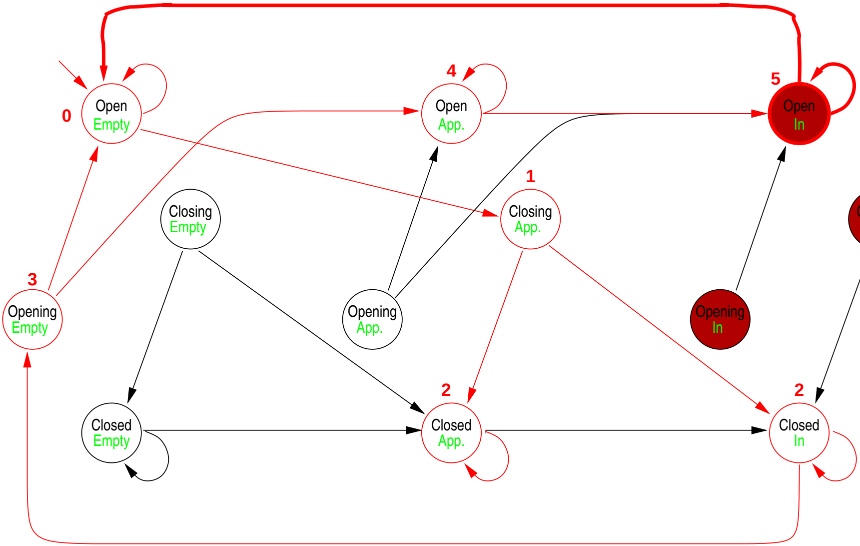


Track model

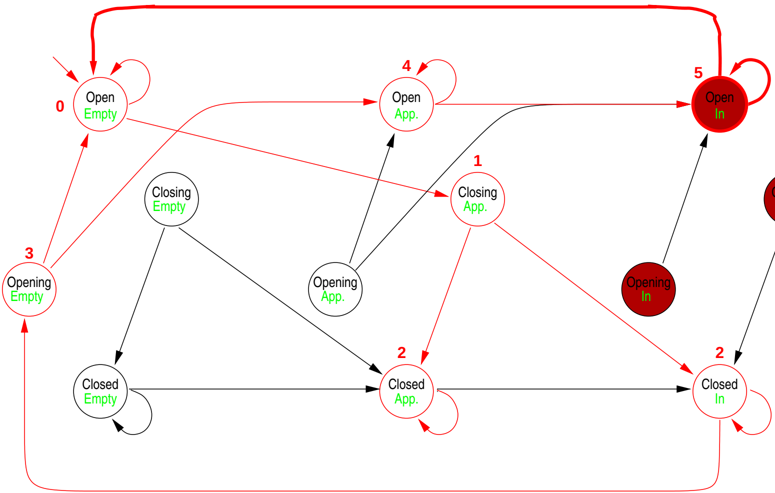
— safe abstraction —



Automatic check



Verification result

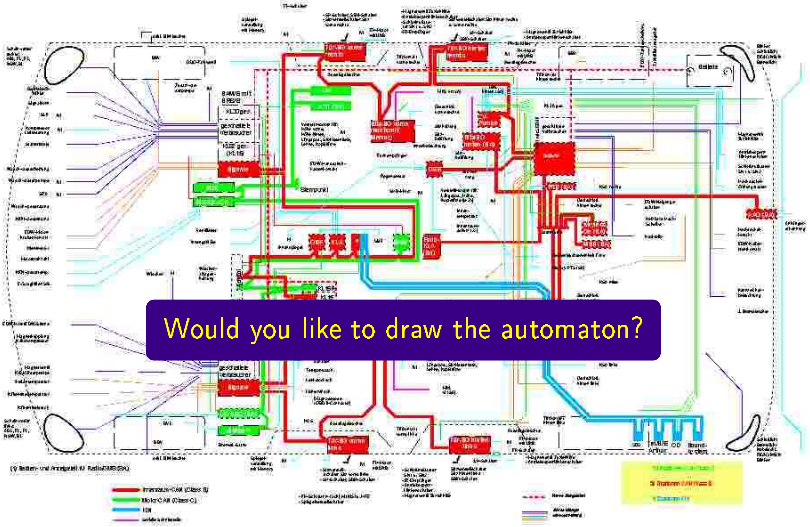


Stimuli:	Empty,	Approach,	In	, Empty	, Approach,	In.
Gate reaction:	Open	, Closing	, Closed,	Opening,	Open	, Open.

Formal Methods vs. Simulation

	Environment dynamics exactly known	Environment dynamics partially known
Simulation	efficient, reasonably exact, yet confined coverage of open systems	at most (rough) approximation, i.g. valid over short time frames only
Formal verific.	full coverage of open systems, at the price of computational complexity	coverage of all possible instances, at the price of computational complexity

Embedded systems in-the-large



Would you like to draw the automaton?

Further problems

- How to obtain the finite-state abstraction of the environment?
- How to justify it?
- How to decide whether a counterexample is real?
  (Could well be an artifact of the abstraction of the environment.)

Let the mass media help us...

"Math is more than formulas or equations; it's logic, it's rationality, it's using your mind to solve the biggest mysteries we know."

Hybrid Systems

What is a hybrid system?

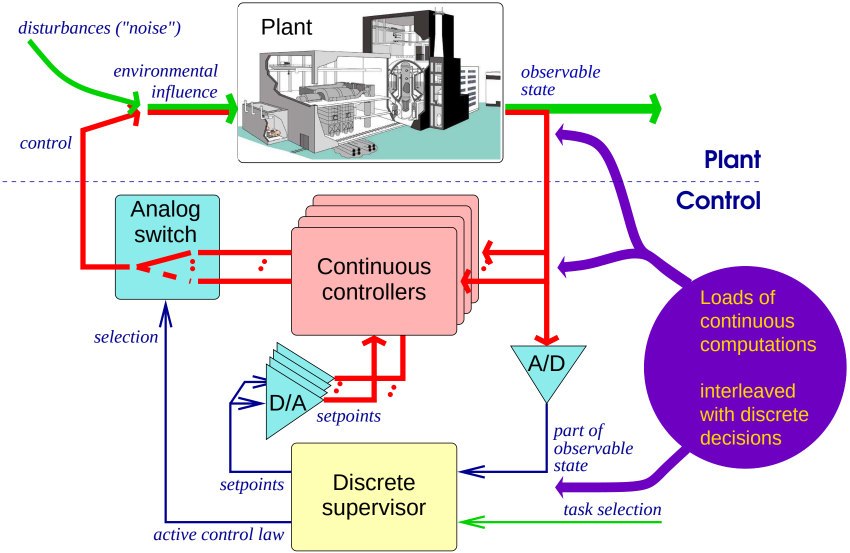
Hybrid (griech.) bedeutet überheblich, hochmütig, vermessen.
Weitere Inhalte [insbes. im wiss. Sprachgebrauch] sind später hinein interpretiert worden.

Hybrid (from Greece) means arrogant, presumptuous.
Other interpretations [in particular, in scientific jargon] have been added later.

After H. Menge: Griechisch/Deutsch, Langenscheidt 1984

$\Rightarrow$ when you try to verify hybrid systems,
be prepared that they may act like a prima donna!

Hybrid Systems



Hybrid systems

are ensembles of interacting discrete and continuous subsystems:

- **Technical systems:**
  - physical plant + multi-modal control
  - physical plant + embedded digital system
  - mixed-signal circuits
  - multi-objective scheduling problems (computers / distrib. energy management / traffic management / ...)
- **Biological systems:**
  - Delta-Notch signaling in cell differentiation
  - Blood clotting
  - ...
- **Economy:**
  - cash/good flows + decisions
  - ...
- **Medicine/health/epidemiology:**
  - infectious diseases + vaccination strategies
  - ...

Discrete vs. continuous

A discrete system

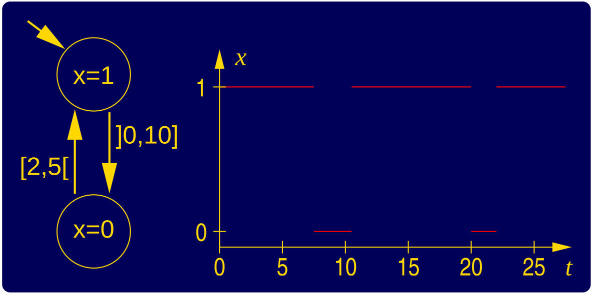
- operates on a state,
- performs discontinuous state changes at discrete time points,
- state is constant in between

E.g., a program

Prog. variables, position

Computation steps: assignments, ctrl. flow

Stable states



Validation by

- **Program verification**
- **State exploration**

Discrete vs. continuous

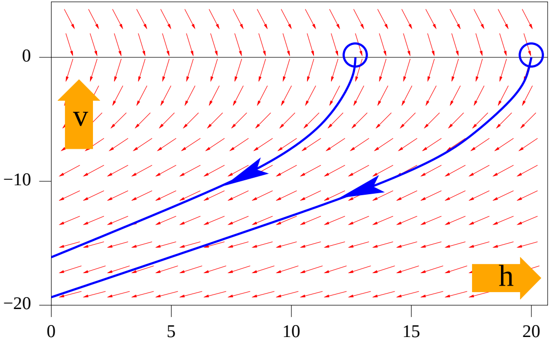
a continuous system

- operates on a continuous state,
- which evolves continuously.

E.g., a ball

Height, speed

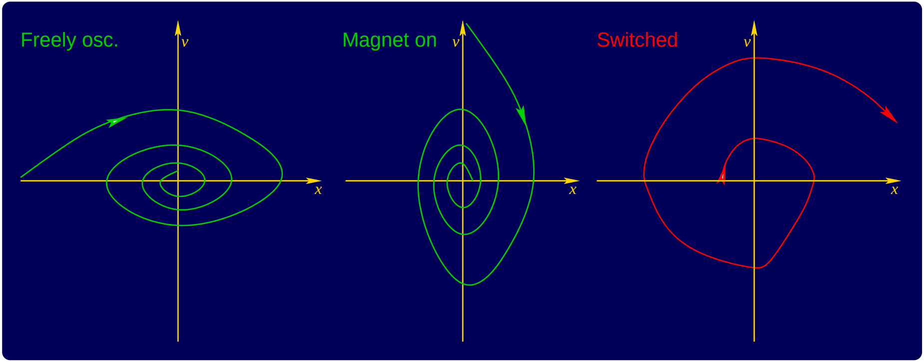
Newtonian mechanics



Validation:

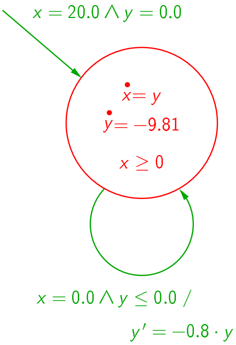
- **Analytically**
- **Simulation + continuity**

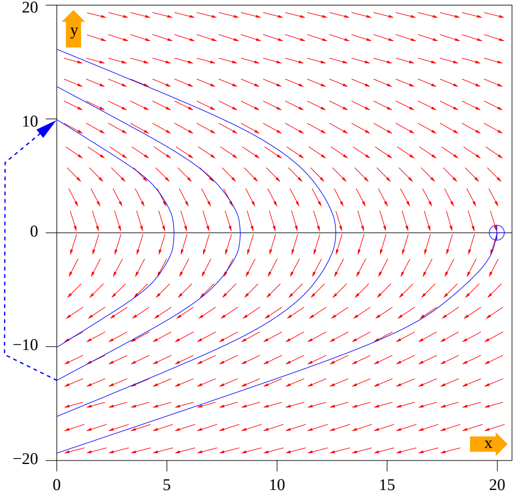
Coupled Dynamics: Forced Pendulum



Interaction of continuous dynamics and discrete mode switch destroys global convergence!

A Formal Model: Hybrid Automata



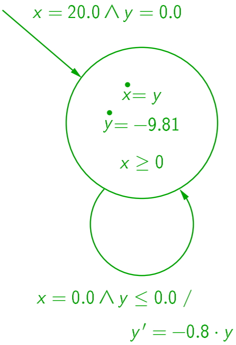


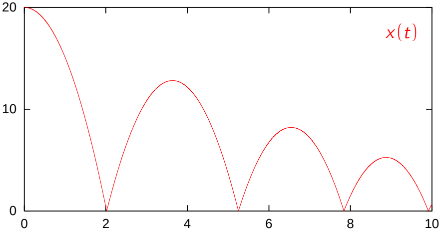
x : vertical position of the ball

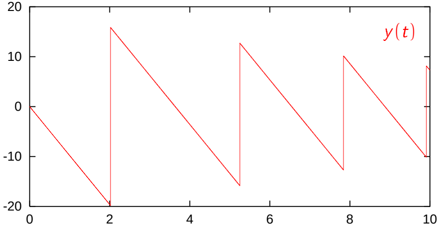
y : velocity

- $y > 0$ ball is moving up
- $y < 0$ ball is moving down

A Formal Model: Hybrid Automata







x : vertical position of the ball

y : velocity

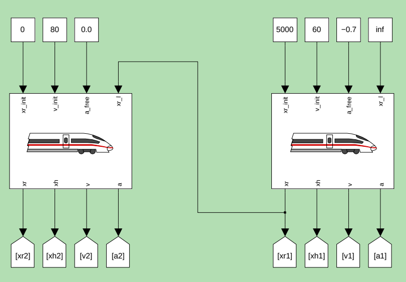
- $y > 0$ ball is moving up
- $y < 0$ ball is moving down

An Example of HS Analysis

Train Separation in ETCS Level 3

BMC of Matlab/Simulink Model

Example: Train Separation in Absolute Braking Distance

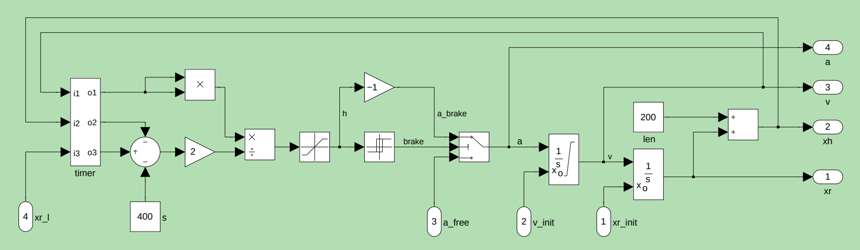


Minimal admissible distance d between two successive trains equals braking distance d_b of the second train plus a safety distance S.

First train reports position of its tail to the second train every 8 seconds.
Controller in second train automatically initiates braking to maintain a safe distance.

Analysis of Matlab/Simulink Model

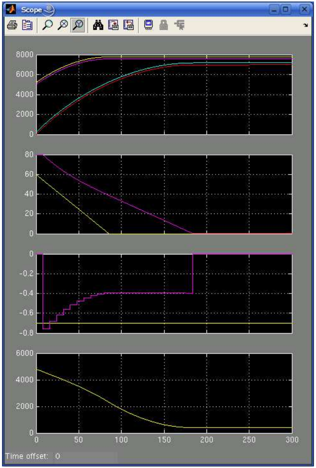
Model of Controller & Train Dynamics



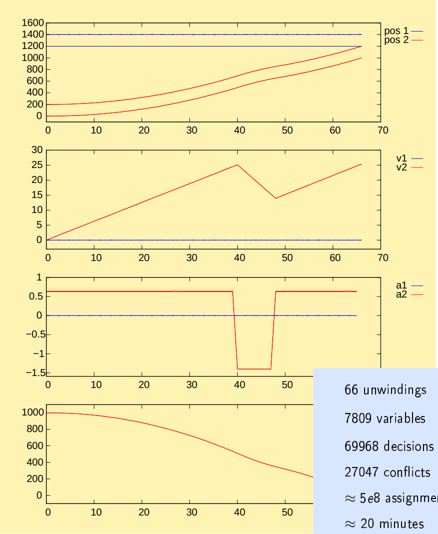
Property to be checked: Does the controller guarantee that collisions don't occur in any possible scenario of use?

Analysis of Matlab/Simulink Model

Simulation of the Model



Error Trace found by HySAT



66 unwindings

7809 variables

69968 decisions

27047 conflicts

$\approx 5e8$ assignments

≈ 20 minutes

The morale

Tausend mal simuliert,
tausend mal ist nix passiert.
Einmal verifiziert,
und es hat "bumm" gemacht.

(a variation on "Tausend mal berührt" by Klaus Lage)

Hybrid Automata

The formal model

Hybrid automata

Hybrid systems	=	Coupled digital & analog systems

$$\downarrow$$

Hybrid automata = *Finite automata* with

- *immediate transitions* that are
- *triggered by predicates on the (continuous) plant state*

+ *evolution of the continuous plant*

- *real-valued variables* governed by
- a set of (restricted) *differential equations* that are
- *selected by the current automaton state*

Hybrid Automaton (w/o input) [after K.H. Johansson]

Def: a hybrid automaton H is a tuple $H = (V, X, f, \mathit{Init}, \mathit{Inv}, \mathit{Jump})$, where :

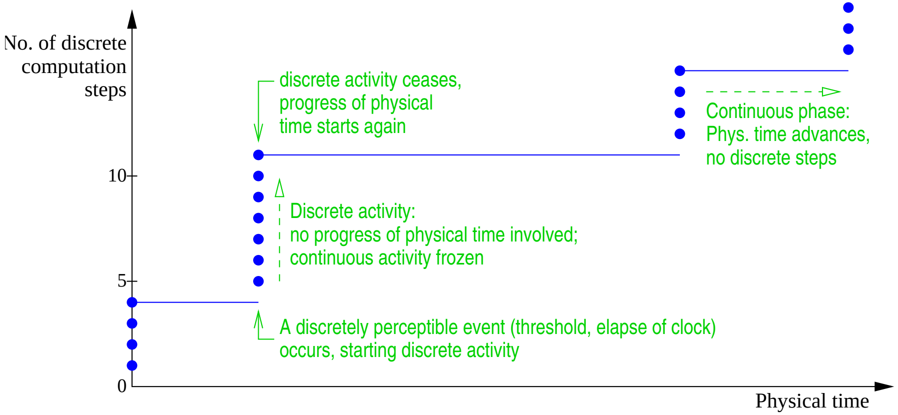
- V is a *finite* set of discrete modes.
  The elements of V represent the discrete states.
- $X = \{x_1, \ldots, x_n\}$ is an (ordered) finite set of continuous variables.
  A real-valued valuation $z \in \mathbb{R}^n$ of $x_1, \ldots, x_n$ represent a continuous state.
- $f \in V \times \mathbb{R}^n \to \mathbb{R}^n$ assigns a vector field to each mode.
  The dynamics in mode m is $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(m, \mathbf{x})$.
- $\mathit{Init} \subseteq V \times \mathbb{R}^n$ is the initial condition.
  Init defines the admissible initial states of H.
- $\mathit{Inv} \subseteq V \times \mathbb{R}^n$ specifies the mode invariants.
  Inv defines the admissible states of H.
- $\mathit{Jump} \in V \times \mathbb{R}^n \to \mathcal{P}(V \times \mathbb{R}^n)$ is the jump relation.
  Jump defines the possible discrete actions of H. The jump relation may be non-deterministic and entails both discrete modes and continuous variables.

Generalizations

This definition of a HA is *not* the most general one. Obvious extensions include

- Input / disturbances in the vector field.
- Labeled jumps.
- Nondeterministic continuous evolutions.
- Stochastic effects.

Semantics: Two-Dimensional Time

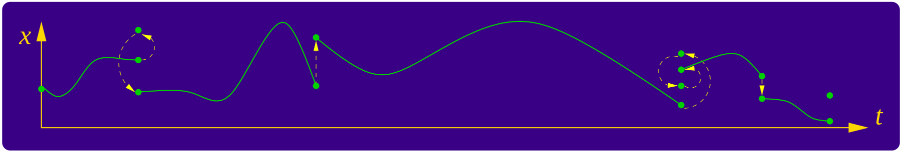


An idealization partially justified by differing speeds of ES and environment!

Hybrid time [after K.H. Johansson]

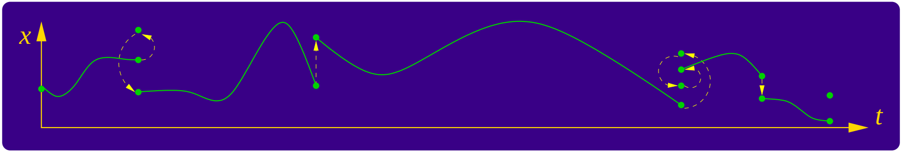
Def: A hybrid time frame is a finite or infinite sequence $\tau = \langle I_1, I_1, \ldots \rangle$ of time intervals I_i, where

- each I_i is a non-empty convex subset of $\mathbb{R}_{\geq 0}$, i.e. a non-empty interval in $\mathbb{R}_{\geq 0}$,
- $\inf I_i \in I_i$ for each i, i.e. the intervals are left-closed,
- $\sup I_i \in I_i$ for each $i < \mathrm{len}\ \tau$, i.e. all intervals excepts perhaps the rightmost are right-closed,
- $\max I_i = \min I_{i+1}$ for each $i < \mathrm{len}\ \tau$, i.e. the intervals are adjacent and overlap exactely in one point.



Def: A hybrid trajectory E is a tuple $E = (\tau, v, x)$ such that

- τ is a hybrid time frame,
- $v \in V^* \cup V^\omega$ with $\operatorname{len} v = \operatorname{len} \tau$ is a sequence of discrete modes,
- $x \in (\mathbb{R}_{\geq 0} \overset{\text{part.,cont.}}{\longrightarrow} \mathbb{R}^n)^* \cup (\mathbb{R}_{\geq 0} \overset{\text{part.,cont.}}{\longrightarrow} \mathbb{R}^n)^\omega$ with $\operatorname{len} x = \operatorname{len} \tau$ and $\operatorname{dom}(x)_i = \tau_i$ is a sequence of continuous flows of the variables in X.

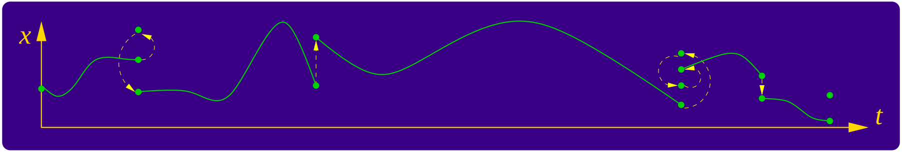


Executions of a HA [after K.H. Johansson]

Def: A run $E = (\tau, v, x)$ is an *execution* of the hybrid automaton $H = (V, X, f, Init, Inv, Jump)$ iff

- **Initiation:** $(v_1, x_1(\min \tau_1)) \in Init$,
- **Consecution:** $Jump((v_i, x_i(\max \tau_i)) \ni (v_{i+1}, x_{i+1}(\min \tau_{i+1}))$ holds for all $i < \mathrm{len}\ \tau$,
- **Continuous evolution:** x_i is a solution of $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(v_i, \mathbf{x})$ for each $i \leq \mathrm{len}\ \tau$,
- **State consistency:** $(v_i, x_i(t)) \in Inv$ for each $t \in \mathrm{dom}(\tau)_i$ and each $i \leq \mathrm{len}\ \tau$

hold.



Types of executions

- An execution (τ, v, x) is finite iff $\mathrm{len}\ \tau < \infty$ and $\tau_{\mathrm{len}\ \tau}$ is bounded.
- An execution (τ, v, x) is infinite iff $\mathrm{len}\ \tau = \infty$ or $\sup \tau_{\mathrm{len}\ \tau} = \infty$.
- An execution (τ, v, x) is a Zeno behavior iff $\mathrm{len}\ \tau = \infty$ but $\sup_{i \in \mathbb{N}} \sup \tau_i < \infty$.
- An execution E of H is maximal iff there is no execution E' of H with $E \preceq E'$ and $E \neq E'$.

Note that the maximal executions of H may *properly* include the infinite executions of H due to possible maximality = non-extensibility of a finite execution ("blocking").

Hybrid systems

- **Proof obligation:** Can the system be guaranteed to show desired behaviour, even under disturbances? E.g.,
  - remains in safe states?
  - eventually reaches a desired operational mode?
  - stabilizes, i.e., converges against a setpoint / stable orbit / region of phase space?

! involves co-verification of controller and *continuous* environment.

State and Dimension Explosion

Number of continuous variables linear in number of cars

- Positions, speeds, accelerations,
- torque, slip, ...

Number of discrete states exponential in number of cars

- Operational modes, control modes,
- state of communication subsystem, ...

Size-dependent dynamics

- Latency in ctrl. loop depends on number of cars due to communication subsystem.
- Coupled dynamics yields long hidden channels chaining signal transducers.

⇒ Need a scalable approach
⇒ Let's try to achieve this through strictly symbolic methods.

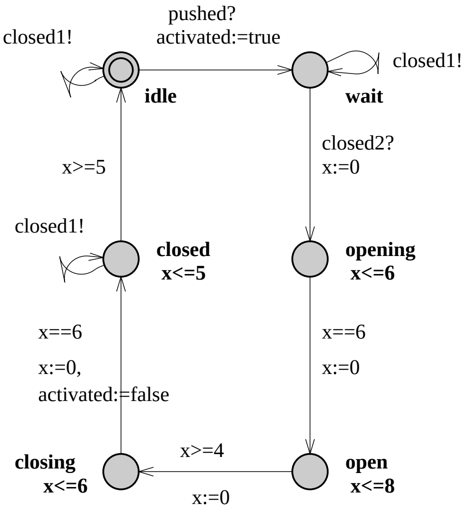
Further Agenda

1. Reachability analysis by iterating the transition relation
   - Decidable cases (very confined...)
     - Timed automata
     - Initialized rectangular automata
   - Decidable wrt. depth-bounded reachability
     - linear hybrid automata
   - Undecidable even in the bounded (the truly interesting stuff...)
     - Non-polynomial discrete-time hybrid automata
     - Non-linear continuous-time hybrid automata
2. Some other verification schemes
   - safe finite-state approximations
   - Lyapunov criteria

Decidable Case

Timed Automata

Example



- Stays in state opening for exactly 6 time units,
- stays in state open between 4 and 8 time units,
- stays in state closing for exactly 6 time units,
- stays in state closed for exactly 5 time units,
- stays in all other states arbitrarily long.

Formal setup

A timed transition system $TTS = (V, E, L, T, \alpha, G, R, Inv, I)$ over a set C of clocks and alphabet Σ has

- a set V of vertices (interpreted as discrete system states, a.k.a. locations),
- a set E of edges (interpreted as possible transitions),
- $L \in V \to \mathcal{P}(AP)$ labels the vertices with atomic propositions that apply in the individual vertices,
- $I \subseteq V$ is a set of initial states,
- $T : E \to (V \times V)$ maps edges to location changes,
- $\alpha : E \to \Sigma$ assigns a communication to transitions,
- $G : E \to \mathcal{P}(ClockVal)$ gives conditions for a transition to be taken,
- $R : E \to \mathcal{P}(C)$ states the clocks to be reset upon a transition,
- $Inv : V \to \mathcal{P}(ClockVal)$ yields state invariants denoting when a state may be held,

where $ClockVal = C \to \mathbb{R}_{\geq 0}$.

Runs of TTS

Given a TTS $(V, E, L, T, \alpha, G, R, Inv, I)$, a run r of the TTS is

- an alternating sequence $(v_0, c_0) \stackrel{(e_0, t_0)}{\longrightarrow} (v_1, c_1) \stackrel{(e_1, t_1)}{\longrightarrow} \ldots$ of
  1. state/clock-valuation pairs $(v_i, c_i) \in V \times ClockVal$,
  2. transition/time pairs $(e_i, t_i) \in E \times \mathbb{R}_{\geq 0}$
- with non-decreasing time stamps: $t_i \leq t_{i+1}$ for each i
- that starts in an initial state: $v_0 \in I$ and $c_0 \equiv 0$
- and is state-transition-consistent: $T(e_i) = (v_i, v_{i+1})$ for each i
- and satisfies the transition guards: $c_i + (t_i - t_{i-1}) \in G(e_i)$ for each i, where $c + t(x) = c(x) + t$ for each clock x and $t_{-1} = 0$,
- and invariably satisfies the state invariants: $c_i + t \in Inv(v_i)$ for each i and each t with $0 \leq t \leq t_i - t_{i-1}$
- and obeys clock resets: $c_{i+1}(x) = \begin{cases} c_i(x) + (t_i - t_{i-1}) & \text{iff } x \notin R(e_i) \\ 0 & \text{iff } x \in R(e_i) \end{cases}$

  for each i and each clock x.

The quest

- The set of states of a TTS is $V \times \mathit{ClockVal}$.
- It is infinite, as $\mathit{ClockVal} = C \to \mathbb{R}_{\geq 0}$.
- Naive forward or backward (on the fly or symbolic) state coloring algorithms need not terminate.

Is reachability analysis etc. nevertheless mechanizable?

Simple clock constraints

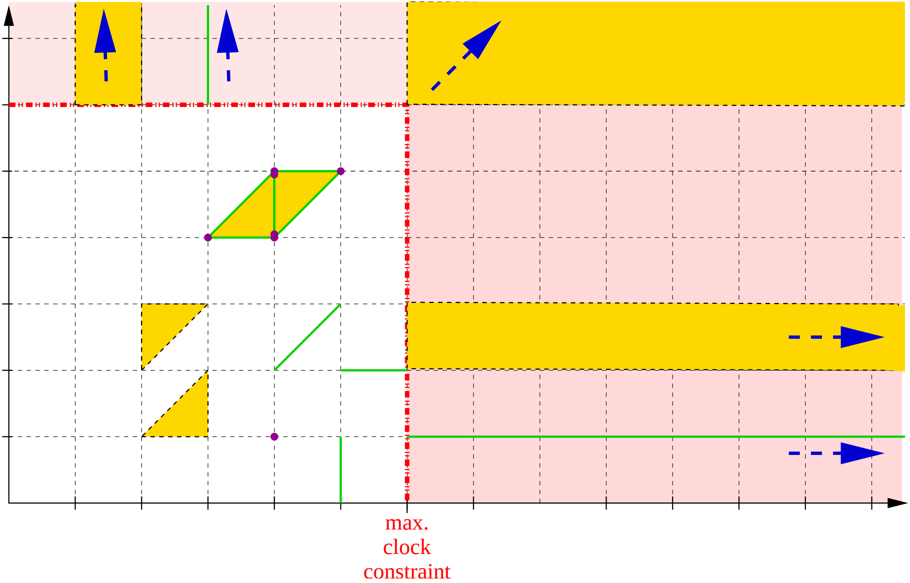
A clock constraint is simple iff

- it is of the form $x \sim k$, where x is a clock, k an integer constant, and $\sim$ one of $<, \leq, =, \geq, >$
- a conjunction of such simple constraints.

From now on, we will concentrate on TTS where

- all guards are simple,
- all invariants are simple.

Clock regions



Time-abstract bisimulation

A time-abstract bisimulation between two TTS is a relation

$$\sim \subset (V \times ClockVal) \times (V' \times ClockVal')$$

s.t. for each $(v, c) \sim (v', c')$:

1. if there is $(v_1, c_1) \in V \times ClockVal$ and $(e, t) \in E \times \mathbb{R}_{\geq 0}$ s.t.

$$(v, c) \stackrel{(e,t)}{\longrightarrow} (v_1, c_1)$$

   then there is $(v_1', c_1') \in V' \times ClockVal'$ and $(e', t') \in E' \times \mathbb{R}_{\geq 0}$ s.t.

$$(v', c') \stackrel{(e',t')}{\longrightarrow} (v_1', c_1') \quad \text{and} \quad \alpha(e) = \alpha(e') \quad \text{and} \quad (v_1, c_1) \sim (v_1', c_1')$$

N.B.: t and t' are not related! $\rightsquigarrow$ time abstraction.

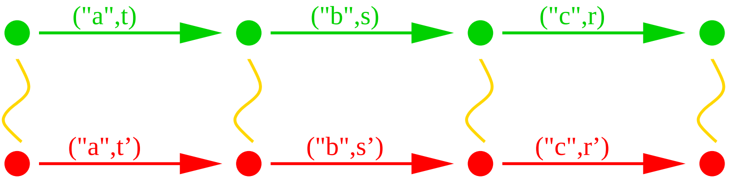
Time-abstract bisimulation (cntd.)

2. if there is $(v_1', c_1') \in V' \times \mathit{ClockVal}'$ and $(e', t') \in E' \times \mathbb{R}_{\geq 0}$ s.t.

$$(v', c') \stackrel{(e',t')}{\longrightarrow} (v_1', c_1')$$

then there is $(v_1, c_1) \in V \times \mathit{ClockVal}$ and $(e, t) \in E' \times \mathbb{R}_{\geq 0}$ s.t.

$$(v, c) \stackrel{(e,t)}{\longrightarrow} (v_1, c_1) \quad \text{and} \quad \alpha(e) = \alpha(e') \quad \text{and} \quad (v_1, c_1) \sim (v_1', c_1')$$



States in the $\sim$ relation follow similar (same labels, different timing) traces.

Clock regions vs. time-abstract bisimulation

Thm.: If $\sim$ is a time-abstract bisimulation *on a TTS* s.t. $\sim$ does only relate identical vertices (yet with potentially different clock val.s) and if $(v, c) \sim (v', c')$ then a vertice $w \in V$ is reachable from (v, c) iff w is reachable from (v', c').

Thm.: For any TTS, the relation $\sim \subset (V \times ClockVal) \times (V \times ClockVal)$ defined by $(v, c) \sim (v', c')$ iff

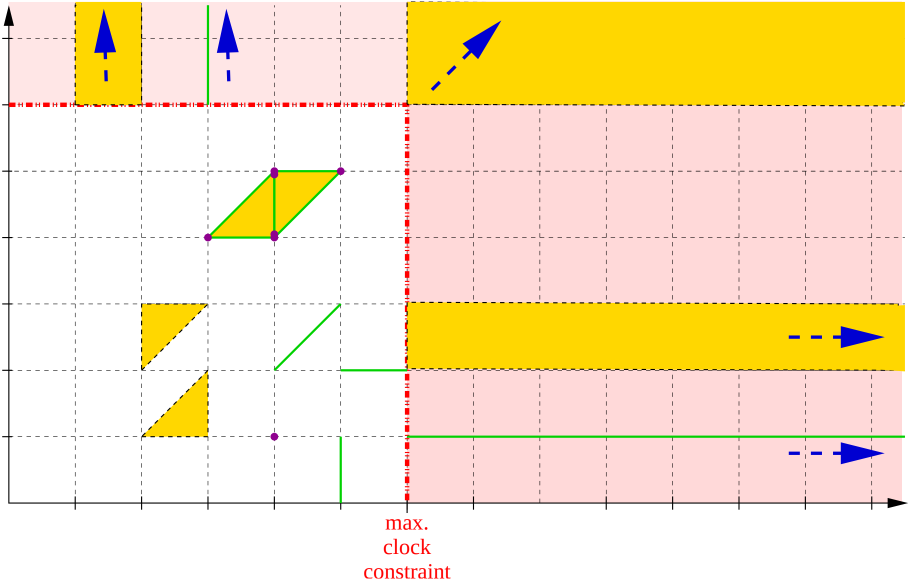
1. $v = v'$,
2. F.e. clock x, $\lfloor c(x) \rfloor = \lfloor c'(x) \rfloor$ or $c(x) > mc < c'(x)$,
3. F.e. clock x, $\mathrm{fract}(c(x)) = 0 \iff \mathrm{fract}(c'(x)) = 0$ or $c(x) > mc < c'(x)$,
4. F.e. clock x, y, $\mathrm{fract}(c(x)) \leq \mathrm{fract}(c(y)) \iff \mathrm{fract}(c'(x)) \leq \mathrm{fract}(c'(y))$ or $\ldots$

is a time-abstract bisimulation on the TTS (i.e., between the states of just that one TTS).
(mc is the maximum time constant in the TTS.)

Cor.: Wrt. vertex reachability (and other time-abstract notions like existence of time-abstract traces), states in the above $\sim$ relation are indistinguishable.

Obs.: For any TTS, there are only finitely many equivalence classes wrt. $\sim$.

Equivalence classes of $\sim$



The region automaton

Given the $TTS = (V, E, L, T, \alpha, G, R, Inv, I)$, we define its region "automaton" (like the TTS, it actually lacks an acceptance condition) to be the finite Kripke structure

$A_{\mathrm{TTS}} = ([V \times ClockVal]_{\sim}, \longrightarrow, L', [I \times \{x \mapsto 0\}]_{\sim})$ with

- $x \longrightarrow y$ iff there is $(v, c) \in x$, $(v', c') \in y$, $t \geq 0$, and $e \in E$ s.t. $(v, c) \overset{(e,t)}{\longrightarrow} (v', c')$
- $L'([v, c]) = L(v)$.

☺ This is a finite Kripke structure that can be subjected to CTL model-checking etc.

☹ but its size is exponential in the number of clocks: $\#\text{regions} = |C|! \cdot 2^{|C|} \cdot \prod_{c \in C}(2 \max(c) + 2)$

? Can we do the state-space traversal more symbolicly, representing *sets* of regions by predicates?

Symbolic Methods for TA

Clock zones

Clock zones

A clock zone is the set of satisfying assignments in $\mathbb{R}^n_{\geq 0}$ of a conjunction of

- inequations that compare a clock to an integer constant and
- inequations that compare the difference of two clocks to an integer constant.

By introduction of a dedicated clock x_0 representing the value 0, difference logic formulae of the specific conjunctive form

$$
\begin{array}{rcl}
\phi & ::= & \bigwedge_{x \in C} (x_0 - x \leq 0) \ \wedge \ \bigwedge_{i=1}^{n} \psi_i \\
\psi_i & ::= & c_{i1} - c_{i2} \sim_i k_i \\
\sim_i & ::= & < \mid \leq \\
k_i & ::\in & \mathbb{Z}
\end{array}
$$

form an appropriate symbolic representation of clock zones.

Closure properties of clock zones

If ϕ and ψ are symbolic representations of clock zones and $d \in \mathbb{N}$ then symbolic representations

- $\phi \wedge \psi$ for zone intersection: $[\![\phi \wedge \psi]\!] \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \mid \mathbf{x} \models \phi \text{ and } \mathbf{x} \models \psi\}$
- $\exists x_i.\phi$ for clock hiding:

$$[\![\exists x_i.\phi]\!] \stackrel{\text{def}}{=} \left\{ (x_1, \ldots, x_n) \;\middle|\; \begin{array}{l} \text{there is } y \in \mathbb{R}_{\geq 0} \text{ s.t.} \\ (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \models \phi \end{array} \right\}$$

- $\phi[x_i := 0]$ for clock reset: $[\![\phi[x_i := 0]]\!] \stackrel{\text{def}}{=} [\![x_i = 0 \wedge \exists x_i.\phi]\!]$
- $\phi \uparrow$ for elapse of time:

$$[\![\phi \uparrow]\!] \stackrel{\text{def}}{=} \{(x_1 + \delta, \ldots, x_n + \delta) \,|\, (x_1, \ldots, x_n) \models \phi, \delta \in \mathbb{R}_{\geq 0}\}$$

can be obtained effectively.

TA reachability using zones: the idea

1. Represent reachable state sets by lists of pairs of locations and clock zones $\langle (l_1, z_1), \ldots, (l_m, z_m) \rangle$,
2. for such a pair, compute the set $Post_t(l, z)$ of successors under a transition t with $T(t) = (l, l')$ by
   - let time elapse starting from z: $\phi_1 = z \uparrow$ represents states reachable under arbitrary passage of time
   - intersect ϕ_1 with $Inv(l)$: $\phi_2 = \phi_1 \wedge Inv(l)$ reflects states reachable through time passage consistent with the location invariant (N.B.: invariant is convex due to simplicity)
   - intersect ϕ_2 with guard $G(t)$: $\phi_3 = \phi_2 \wedge G(t)$ reflects states reachable through time passage which enable the transition t
   - reset the clocks in $R(t)$: $\phi_4 = \phi_3[r_1 := 0] \ldots [r_j := 0]$, where $\{r_1, \ldots, r_j\} = R(t)$, reflects the clock readings after performing t's resets
   - intersect with the target loc.'s invariant: $\phi_5 = \phi_4 \wedge Inv(l')$
   - do the location change: $Post_t(l, z) = (l', \phi_5)$.

The state-space exploration

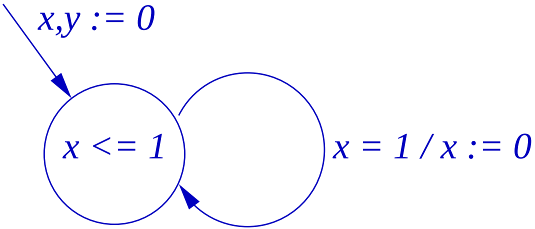
1. Start with the state list
   $R_0 = I \times \{\text{``}\bigwedge_{x \in C}(x_0 - x \leq 0) \wedge \bigwedge_{x \in C}(x - x_0 \leq 0)\text{''}\}$.
2. Repeat
   1. select $(l_i, z_i) \in R_k$ and $t \in E$ with source l_i s.t. $Post_t(l_i, z_i)$ is not already subsumed by R_k,
   2. build $R_{k+1} = R_k \cdot \langle Post_t(l_i, z_i) \rangle$

   until no such $(l_i, z_i) \in R_k$ and $t \in E$ is found.

N.B. Subsumption test can be performed at various levels of detail.

The problem

- Iterating $Post_t(l, z)$ for all pairs (l, z) in the list of reachable states and all transitions need not terminate:



- In the region graph, we solved the problem by not distinguishing clock readings above the max. clock constant.
- We can achieve a similar effect by widening zones that extend beyond the max. clock constant:
    - Any constraint of the form $x_i - x_j \sim l$ with $l > maxconstant$ is removed from the symbolic representation when it arises.

Manipulation of Difference Logic Constraints

Difference Bound Matrices

Difference Bound Matrices

Difference bound matrices (DBMs) are a canonizable representation for *conjunctive* formulae in difference logic

$$
\begin{array}{rcl}
\phi & ::= & \bigwedge_{i=1}^{n} \psi_i \\
\psi_i & ::= & c_{i1} - c_{i2} \sim_i k_i \\
\sim_i & ::= & < \mid \leq \\
k_i & ::\in & \mathbb{Z}
\end{array}
$$

Given a finite clock set C (in practice containing the pseudo-clock x_0), a DBM M over C is a mapping

$$\underbrace{(C \times C)}_{\text{clock pairs}} \rightarrow (\underbrace{\{<, \leq\} \times \mathbb{Z}}_{\text{constraint on diff.}} \cup \underbrace{\{(<, \infty)\}}_{\text{unconstrained}}) .$$

Encoding: $M(x, y) = (\sim, k) \;\triangleq\; x - y \sim k$

Implied constraints and tightening

Observation: $x - y \sim_1 k_1$ and $y - z \sim_2 k_2$ implies $x - z \sim k_1 + k_2$, where

$$\sim = \begin{cases} \sim_1 & \text{iff } \sim_1 = \sim_2 \\ < & \text{otherwise.} \end{cases}$$

Consequence: A DBM may contain constraint pairs which imply constraints that are tighter than the recorded constraints: $M(x,y) = (\sim_1, k_1) \wedge M(y,z) = (\sim_2, k_2) \wedge M(x,z) = (\sim, k)$ and

1. $k > k_1 + k_2$ or
2. $k = k_1 + k_2$ but $\sim = \leq$, yet $\sim_1 = <$ or $\sim_2 = <$.

Solution: *Tighten the DBM* by replacing the constraint by the stronger implied constraint.
Repeat this until no implied constraint stronger than a recorded constraint remains. This brings the DBM into a *canonical form*.
Such canonization of DBMs can be done in cubic time using the *Floyd-Warshall algorithm*.

Properties of canonical DBMs

Thm: A *canonical* DBM is unsatisfiable iff there is some $x \in C$ such that $M(x, x) = (<, 0)$ or $M(x, x) = (\sim, k)$ with $k < 0$.

Cor: Satisfiability test of *canonical* DBMs runs in $O(|C|)$ time.

Operations on clock zones using canon. DBMs

Intersection:

$$M \wedge N(x,y) = \begin{cases} M(x,y) & \text{if } M(x,y) \text{ is tighter than } N(x,y) \\ N(x,y) & \text{otherwise} \end{cases}$$

Clock reset: When the dedicated clock variable x_0 is used,

$$M[z := 0](x,y) = \begin{cases} M(x,y) & \text{if } x \neq z \text{ and } y \neq z \\ M(x,x_0) & \text{if } x \neq z \text{ and } y = z \\ M(x_0,y) & \text{if } x = z \text{ and } y \neq z \\ (\leq, 0) & \text{if } x = y = z \end{cases}$$

Note that canonicity saves an explicit quantifier elimination as the implied constraints are already in place!

These operations do not preserve canonicity!

Operations on clock zones using canon. DBMs

Elapse of time: When the dedicated clock variable x_0 is used,

$$M \uparrow (x,y) = \begin{cases} M(x,y) & \text{if } x = x_0 \text{ or } \quad y \neq x_0 \\ (<, \infty) & \text{if } x \neq x_0 \text{ and } y = x_0 \end{cases}$$

Widening: When the maximum clock constant is k,

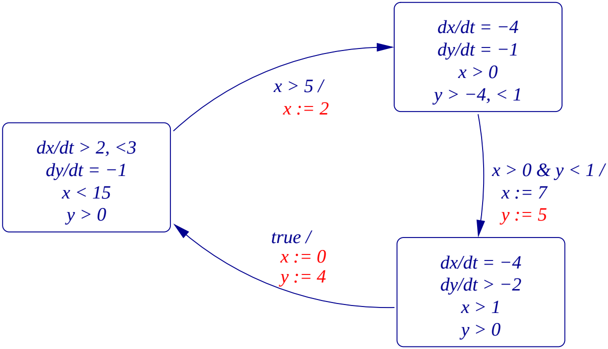
$$\widetilde{M}(x,y) = \begin{cases} M(x,y) & \text{if } M(x,y) = (\sim, l) \text{ with } |l| \leq |k| \\ (<, \infty) & \text{otherwise} \end{cases}$$

This technology

- is implemented in mature tools
  - KRONOS [M. Bozga, C. Daws, O. Maler, A. Olivero, S. Tripakis, S. Yovine, 1994–2002]
  - UPPAAL [K.G. Larsen, Wang Yi, G. Behrmann, P. Pettersson, A. David, B. Nielsen, A. Skou, J. Hąnsson, J.I. Rasmussen, P. Krcal, U. Larsen, M. Mikucionis, L. Mokrushin, T. Amnell, J. Bengtsson, E. Fersman, E. Fleury, T. Hune, K.J. Kristoffersen, F. Larsson, D. Lime, O. Möller, J. Pearson, C. Weise, R.G. Madsen, S.K. Mortensen, 1995–]
- extends to some slightly more hybrid domains:
  - Linear Priced Timed Automata [Larsen, Rasmussen, Boyer, Brihaye, Bruyère, Raskin, 2005–]
    - Locations and transitions come equipped with costs / cost rates charged for taking the transition / staying in the location
    - Guards and invariants may not query cost variables
    - Problem is to determine infimum cost for reaching some state
  - Initialized rectangular hybrid automata [Henzinger, Kopke, Puri, Varaiya, 1995]
    - Rectangular guards, invariants and piecewise constant, rectangular differential inclusions
    - Continuous variables reset whenever differential inclusion changes

Initialized Rectangular Hybrid Automata

Initialized Rectangular Hybrid Automata



Hybrid automata subject to the following restrictions:

- Being initialized: Continuous variables are reset whenever the pertinent differential inclusion changes
- Being rectangular:
  - guards and invariants are simple,
  - continuous evolution is governed by (location-dependent) constant differential inclusions.

Reduction to Timed Automata (Sketch)

1. Replace each variable x by two copies x_l and x_u:
   - x_l represents lower, x_u upper bound on value
   - replace differential inclusions by DEs accordingly:

   $$-2 \leq \frac{\mathrm{d}x}{\mathrm{d}t} \leq 1 \qquad \rightsquigarrow \qquad \frac{\mathrm{d}x_l}{\mathrm{d}t} = -2 \wedge \frac{\mathrm{d}x_u}{\mathrm{d}t} = 1$$

   - adjust guards and invariants accordingly.
2. Remove all DEs of form $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$:
   - due to initialization, value upon entering the assoc. location is known,
   - thus, the respective guards and invariants can be evaluated statically.
3. Scale all DEs to form $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$:
   - Replace DE $\frac{\mathrm{d}y}{\mathrm{d}t} = a$ by $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$;
   - multiply constants in the respective guards and invariants by $\frac{1}{a}$,
   - revert inequality signs iff $a < 0$.
4. Make all constants in guards and invariants integer by multiplying with common denominator (doesn't affect reachability).

Linear Priced Timed Automata

Linear Priced Timed Automata

...extend the concept of timed automata by a monotonicly increasing, linear cost function:

- each transition e is associated a cost $P(e) \in \mathbb{N}$,
  this cost is charged whenever the transition is taken,
- each location v is associated a cost rate $P(v) \in \mathbb{N}$
  this cost rate is charged for staying in the location,
  i.e. staying in v for δ time units costs $P(v) \cdot \delta$

Costs accumulate over a run of the TA, yet cannot be queried by the LPTA:

- the cost of a run is the sum of all costs charged along the run, i.e. by its transitions including the delay transitions,
- the cost accumulated so far cannot be queried in guards or invariants,
- neither can it be updated by any other than the above cost accumulation mechanism.

Formal setup

A linear priced timed automaton $A = (V, E, L, T, \alpha, G, R, Inv, I, P)$ over a set C of clocks and alphabet Σ is composed of

- a timed automaton $A' = (V, E, L, T, \alpha, G, R, Inv, I)$ and
- a cost function $P : (V \cup E) \to \mathbb{N}$.

A run r of A is an alternating sequence

$$r = (v_0, c_0) \stackrel{(e_0, t_0)}{\longrightarrow} (v_1, c_1) \stackrel{(e_1, t_1)}{\longrightarrow} \dots (v_n, c_n)$$

of state/clock-valuation pairs $(v_i, c_i) \in V \times \mathit{ClockVal}$ such that r is a run of the TA A'.

The price $p(r)$ of run r is the accumulated cost

$$p(r) = \sum_{i=0}^{n-1} P(e_i) + \sum_{i=0}^{n-1} (t_i - t_{i-1}) \cdot P(v_i)$$

where $t_{-1} = 0$.

The goal

Given an LPTA $A = (V, E, L, T, \alpha, G, R, Inv, I, P)$ and a location $g \in V$ (or a set $G \subseteq V$ of locations),

1. determine $mincost = \inf\{p(r) \mid r \text{ is a run of } A \text{ ending in } g\}$,
2. if $mincost < \infty$ then find a run r ending in g with $p(r) = mincost$.

This is called the minimum cost reachability problem.

Generalizing zone-based analysis to LPTA

Idea: Extend the concept of a zone into a priced zone

- consists of zone Z, i.e. a set of clock valuations,
- plus cost information ci for *each* clock valuation in the zone.

Usage: Manipulate such priced zones in foward reachability analysis:

- use forward reachability to collect the reachable state in lists of pairs (v, pz), where $v \in V$ and pz is a priced zone,
- for each clock valuation cv in pz, the price information records the *greatest lower bound* of the cost for reaching (v, cv).

Requires: Effective manipulation of priced zones:

- intersection with a (unpriced) zone: for guards and invariants,
- reset operations $pz[x := 0]$ on priced zones,
- closure elapse of time $pz \uparrow$ on priced zones.

Priced zone

Def.: A priced zone is a tuple (Z, c, r), where

- Z is a zone,
- $c \in \mathbb{N}$ denotes the infimum of the price of all clock valuations in the zone, called offset of the zone,
- $r : C \to \mathbb{Z}$ assigns a cost rate to each clock.

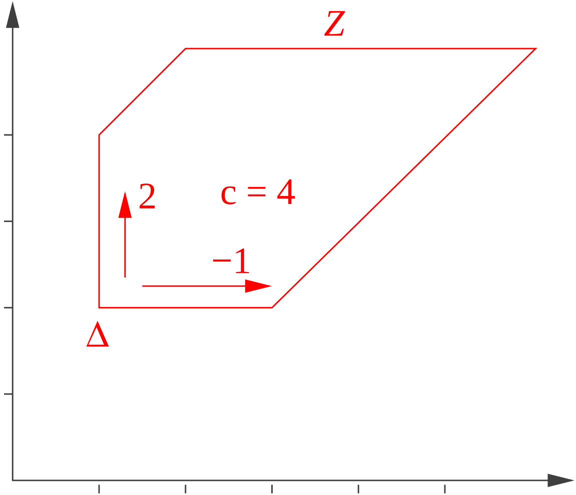
Def.: The cost associated to a clock valuation $cv \in Z$ wrt. the priced zone (Z, c, r) is

$$c + \sum_{x \in C} r(x) \cdot (cv(x) - \Delta_Z(x))$$

where $\Delta_Z(x) = \inf\{cv(x) \mid x \in Z\}$ f.e. $x \in C$ is the "lowermost corner" of Z.

Linear dependency of cost on clock valuation sufficient?

Priced zones



$$\text{cost} = 2y - x + 1$$

Infimum cost of a priced zone

As the cost associated cost of a clock valuation *cv* in the priced zone (Z, c, r) is

$$c + \sum_{x \in C} r(x) \cdot (cv(x) - \Delta_Z(x)),$$

the infimum cost $infcost(Z, c, r)$ of priced zone (Z, c, r) is

$$\inf \left\{ c + \sum_{x \in C} r(x) \cdot (cv(x) - \Delta_Z(x)) \,\middle|\, cv \in Z \right\}$$

Domination of a priced zone by another

We say that priced zone (Z, c, r) is dominated by priced zone (Z', c', r'), denoted $(Z, c, r) \sqsubseteq (Z', c', r')$, iff

- Z' covers Z, i.e. $Z \subseteq Z'$, and
- the prices assigned by (Z', c', r') are cheaper, i.e.

$$c + \sum_{x \in C} r(x) \cdot (cv(x) - \Delta_Z(x)) \geq c' + \sum_{x \in C} r'(x) \cdot (cv(x) - \Delta_{Z'}(x))$$

for each $cv \in Z$.

The cost-aware forward reachability algorithm

$Cost := \infty$
$Passed := \emptyset$
$Waiting := \{(i, (Z, c, r)) \mid i \in I, Z = \bigwedge_{x \in C} c - x_0 = 0, c = 0, r \equiv 0\}$
while $Waiting \neq \emptyset$ *do*
  $(v, (Z, c, r)) :\in Waiting;\ Waiting := Waiting \setminus \{(v, (Z, c, r))$
  if $v = g$ *then* $Cost := \min(Cost, infcost(Z, c, r))$ *fi*
  if $\neg\exists(v, (Z', c', r')) \in Passed \bullet (Z, c, r) \sqsubseteq (Z', c', r')$ *then*
    $Passed := Passed \cup \{(v, (Z, c, r))\}$
    $Waiting := Waiting \cup \{Post_t((v, (Z, c, r))) \mid t \in E\}$
  fi
return $Cost$

[G. Behrmann, 2004]

Symbolic operations

To compute $Post_t((v,(Z,c,r)))$ in forward reachability, we need

- increment of cost by a constant,
- intersection with a zone,
- reset operations $pz[x := 0]$ on priced zones,
- closure elapse of time $pz\uparrow$ on priced zones.

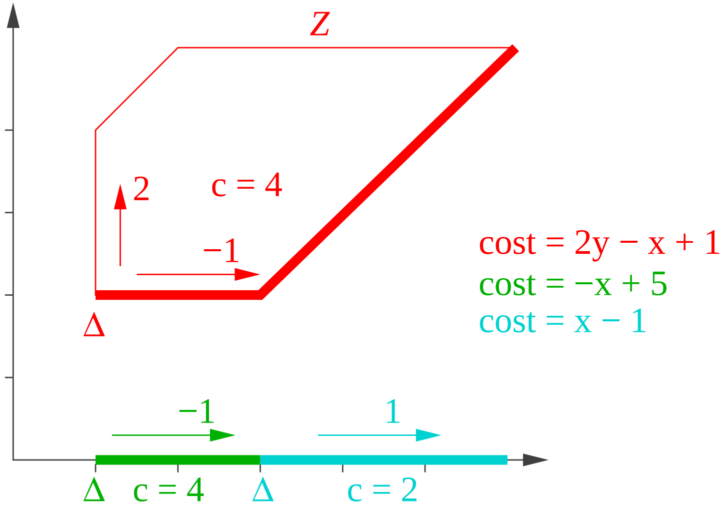
Increment: Given a priced zone (Z,c,r) and aconstant $k \in \mathbb{N}$, the zone's cost increment by k is the priced zone $(Z, c+k, r)$.

Intersection: Given a priced zone (Z,c,r) and an unpriced zone Y, the intersection is

$$(Z \wedge Y, \underbrace{c + \sum_{x\in C} r(x)\cdot(\Delta_{Z\wedge Y}(x) - \Delta_Z(x))}_{\text{cost of the new ``lowermost corner''}}, r)$$

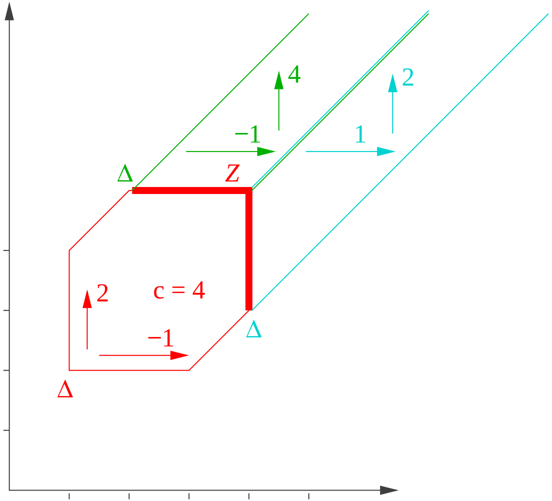
which is a priced zone that can be built effectively from (Z,c,r) and Y.

Priced zones: $y := 0$



Priced zone has to be split!

Priced zones: passage of time



Priced zone has to be split!

Discussion

- Technology is available in UPPAAL CORA [G. Behrman, K.G. Larsen, J.I. Rasmussen, 2002–]
- Various extensions to base algorithm discussed in literature:
  - Multiple cost variables [Larsen, Rasmussen, 2005]
  - Positive and negative costs, infimal and supremal costs [Boyer, Brihaye, Bruyère, Raskin 2007]
- Yet not sufficiently expressive to model feedback between controller and environment:
  - the cost accumulated so far cannot be queried in guards or invariants,
  - neither can it be updated by any other than the accumulation mechanisms.