

# Implementation of First-Order Theorem Provers

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# First-Order Theorem Proving

**Given:** A set axioms and a hypothesis in first-order logic

$$A = \{A_1, \dots, A_n\}, H$$

**Question:** Do the axioms logically imply the hypothesis?

$$A \stackrel{?}{\models} H$$

An **automated theorem prover** tries to solve this question!

# First-Order Logic with Equality

- ▶ First order logic deals with
  - Elements
  - Relations between elements
  - Functions over elements
  - . . . and their combination
- ▶ Allows general statements using quantified variables
  - There exists an  $X$  so that property  $P$  holds ( $\exists X : P(X)$ )
  - For all possible values of  $X$  property  $P$  holds ( $\forall X : P(X)$ )
- ▶ Function and predicate symbols are **uninterpreted**
  - No **implicit** background theory
  - All properties have to be specified explicitly
  - Exception: Equality is interpreted (as a congruence relation)

# Why First-Order Logic?

- ▶ Expressive:
  - Can encode any computable problem
  - Most tasks can be specified reasonably naturally
  - Many other logics can be reasonably translated to first-order logic
- ▶ Automatizable:
  - Sound and complete calculi for proof search exist
  - Search procedures are reasonably efficient
- ▶ Stable:
  - Logic is well-known and well-understood
  - Semantics are clear (and somewhat intuitive)

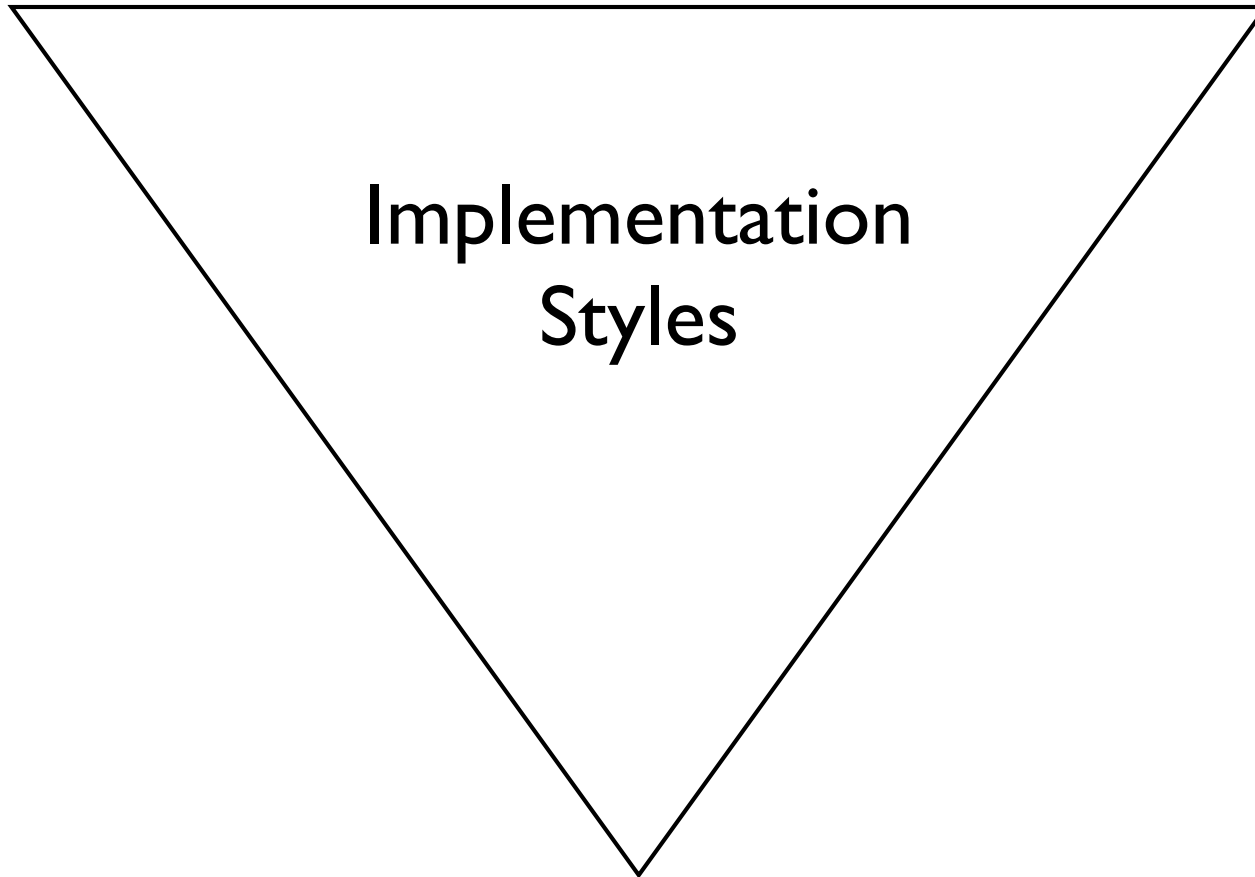
First-order logic is a good compromise between expressiveness and automatizability

# Mainstream Milestones

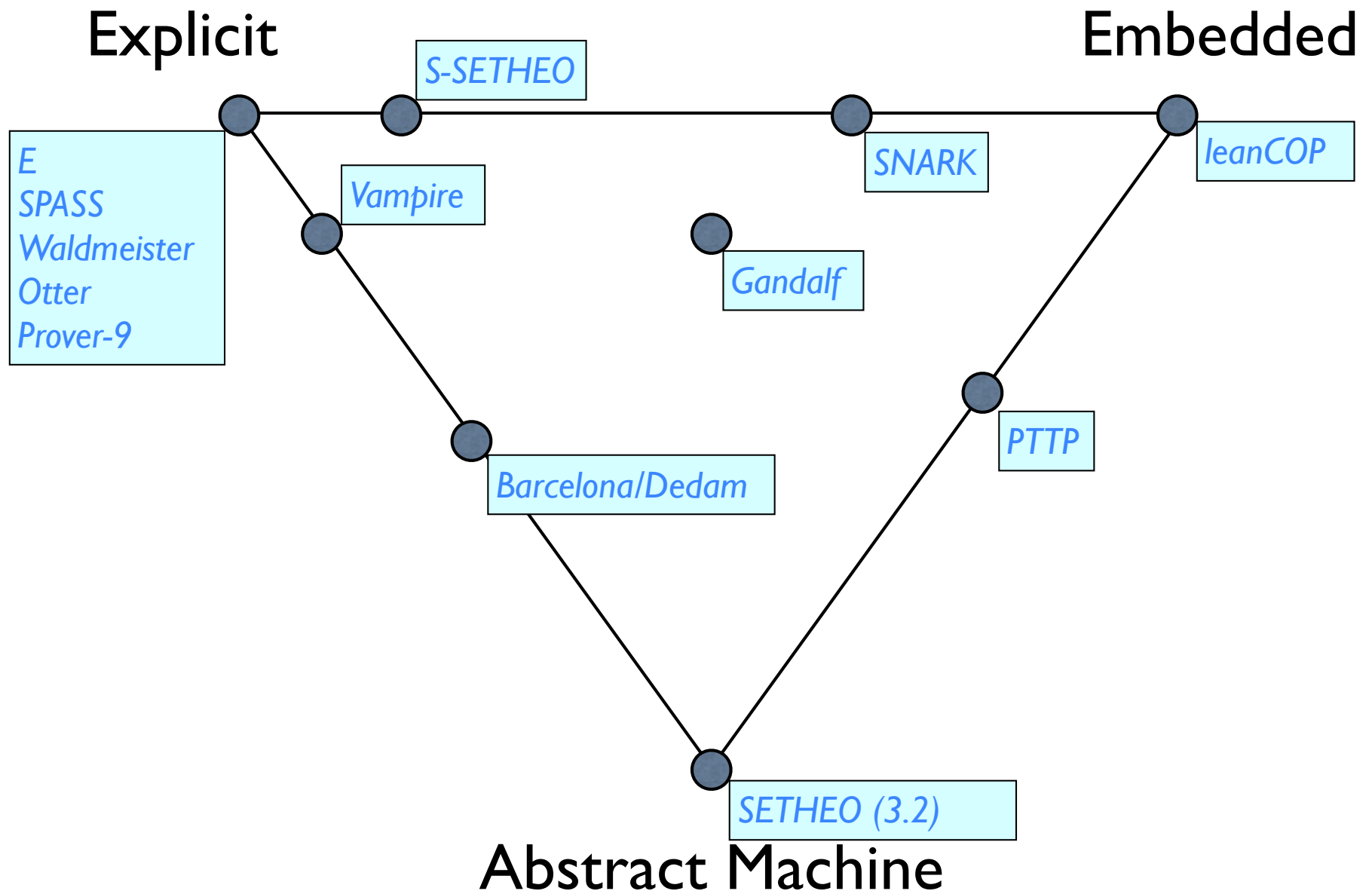
- Herbrand-Universe Enumeration+SAT [DP60]
- Resolution [Rob65]
- Model Elimination [Lov68]
- Paramodulation [RW69]
- Completion [KB70]
- Otter 1.0 (1989, McCune)
- Unfailing completion [BDP89, HR87]
- Superposition [BG90, NR92, BG94]
- SETHEO [LSBB92]
- Vampire [Vor95] (but kept hidden for years)
- First CASC competition at Rutgers, FLOC'96 (Sutcliffe, Suttner)
- Waldmeister [BH96]
- SPASS [WGR96]
- E [Sch99]

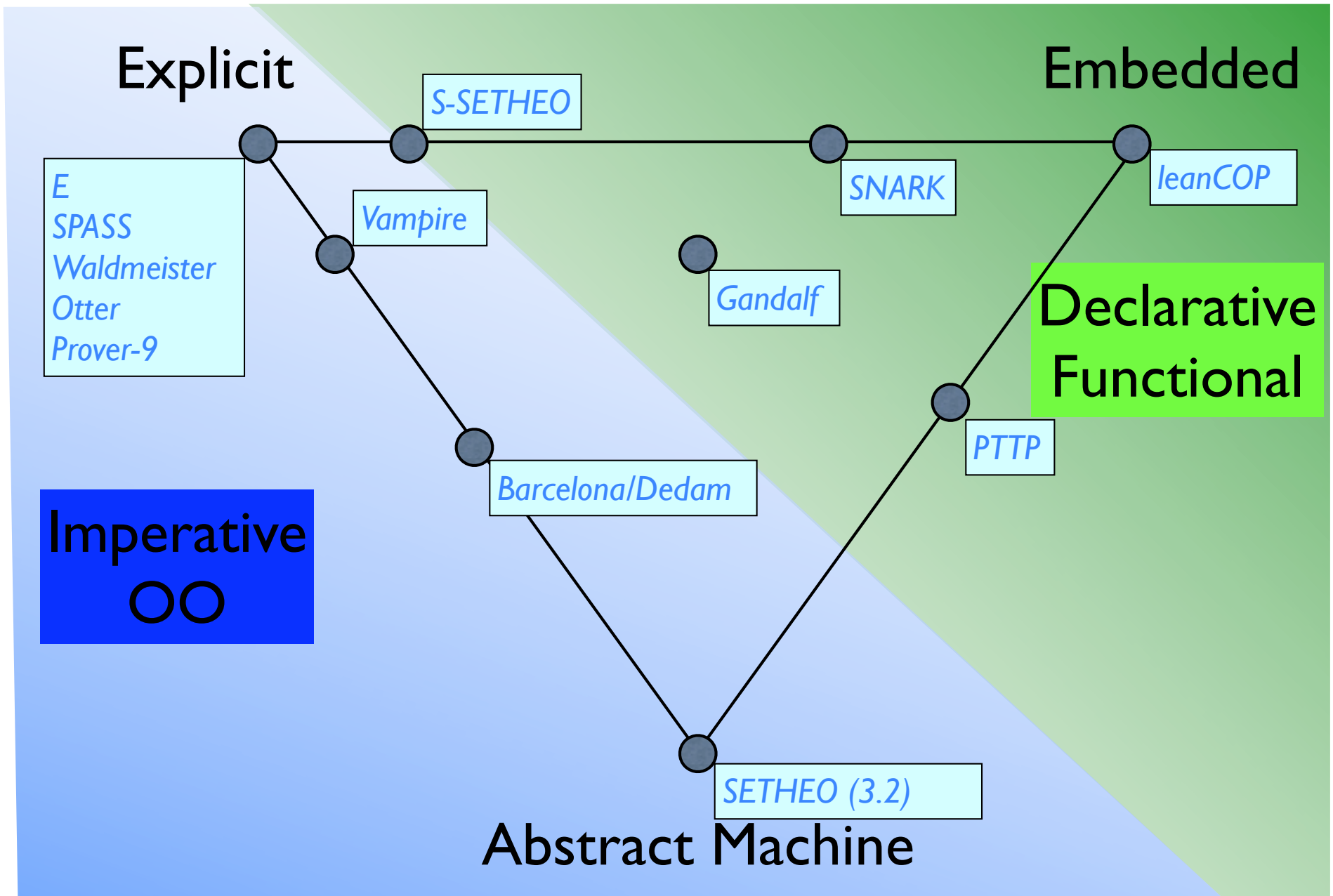
**Explicit**

**Embedded**



**Abstract Machine**







## Implementation Style (References)

|                                    |                                      |
|------------------------------------|--------------------------------------|
| Barcelona/Dedam [NRV97]            | E [Sch02, Sch04b]                    |
| Gandalf [Tam97]                    | Otter [MW97]                         |
| PTTP [Sti92, Sti89]                | Prover-9 [McC08]                     |
| S-SETHEO [LS01b]                   | SETHEO [LSBB92, MIL <sup>+</sup> 97] |
| SPASS [Wei01, WSH <sup>+</sup> 07] | Snark [E.S08]                        |
| Vampire [RV02]                     | Waldmeister [LH02, GHLS03]           |
| leanCOP [OB03, Ott08]              |                                      |

# Formulae

- ▶ Formulas are recursively defined:
  - Literals (elementary statements) are formulae
  - If  $F$  is a formula,  $\forall X : F$  and  $\exists X : F$  are formulae
  - Boolean combinations of formulae are formulae
  - Parentheses are applied wherever necessary
  
- ▶ Example:
  - $\forall X : (\forall Y : ((\text{odd}(X) \wedge \text{odd}(Y)) \rightarrow X \neq \text{add}(Y, 1)))$

# Clauses

- ▶ Clauses are multisets written and interpreted as disjunctions of literals
  - All variables implicitly universally quantified

- ▶ Example:

$$X \neq \text{add}(Y, 1) \vee \text{odd}(X) \vee \text{odd}(Y)$$

- ▶ Alternative views: Implicational

$$X \simeq \text{add}(Y, 1) \implies (\text{odd}(X) \vee \text{odd}(Y))$$

or

$$(X \simeq \text{add}(Y, 1) \wedge \neg \text{odd}(X)) \implies \text{odd}(Y)$$

or

$$(X \simeq \text{add}(Y, 1) \wedge \neg \text{odd}(Y)) \implies \text{odd}(X)$$

or (weirdly)

$$(\neg \text{odd}(Y) \wedge \neg \text{odd}(X)) \implies X \neq \text{add}(Y, 1)$$

# Literals

- ▶  $X \not\approx add(Y, 1) \vee odd(X) \vee odd(Y)$
- ▶ –  $X \not\approx add(Y, 1)$  is a negative equational literal  
–  $odd(X)$  and  $odd(X)$  are positive non-equational literals
- ▶ Conventions:
  - $s \not\approx t$  is a more convenient way of writing  $\neg s \simeq t$
  - We write  $s \simeq t$  to denote an equational literal that may be either positive or negative
  - $s \simeq t$  is a more convenient way of writing  $\simeq (s, t)$

# Literals

- ▶  $X \neq add(Y, 1) \vee odd(X) \vee odd(Y)$
- ▶ –  $X \neq add(Y, 1)$  is a negative equational literal  
–  $odd(X)$  and  $odd(X)$  are positive non-equational literals
- ▶ Convention:
  - $s \neq t$  is a more convenient way of writing  $\neg s \simeq t$
  - We write  $s \simeq t$  to denote an equational literal that may be either positive or negative
  - **Heresy:**  $s \simeq t$  is a more convenient way of writing  $\simeq (s, t)$
  - **Truth:**  $odd(X)$  is a more convenient way of writing  $odd(X) \simeq \top$

# Equational Encoding Snag

## ▶ Problem:

- $\{X \simeq a, \neg p(a)\}$  is satisfiable
- What about  $\{X \simeq a, p(a) \neq \top\}$ ?

## ▶ Solution:

- Two sorts: Individuals and Booleans
- Variables range over individuals only
- Predicate terms are sort Bool

## ▶ Implemented that way in E

# Terms

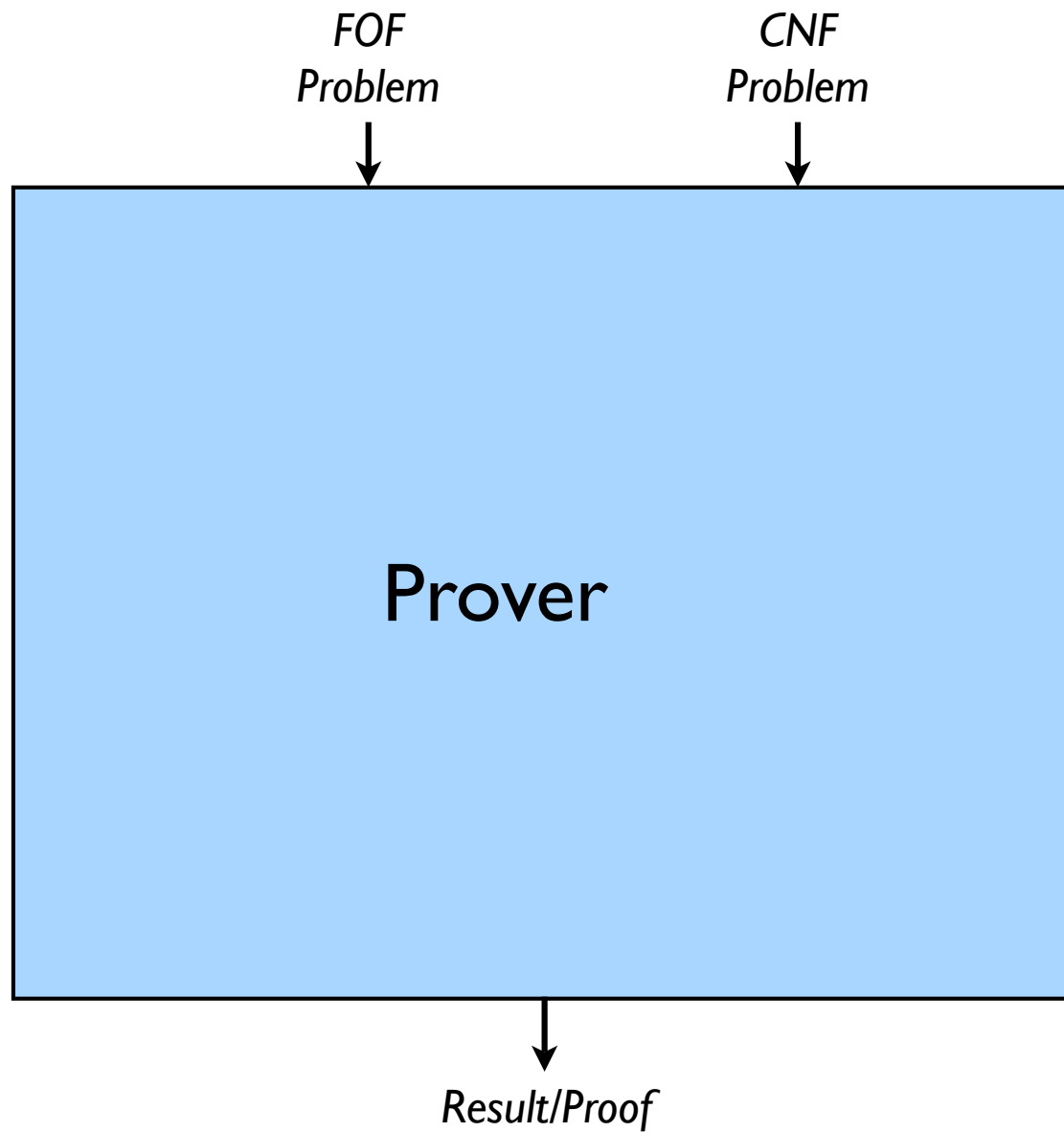
- ▶  $X \neq add(Y, 1) \vee odd(X) \vee odd(Y)$
- ▶ –  $X$ ,  $add(Y, 1)$ ,  $1$ , and  $Y$  are **terms**
- $X$  and  $Y$  are **variables**
- $1$  is a **constant term**
- $add(Y, 1)$  is a **composite term** with proper **subterms**  $1$  and  $Y$

# Concrete Syntax

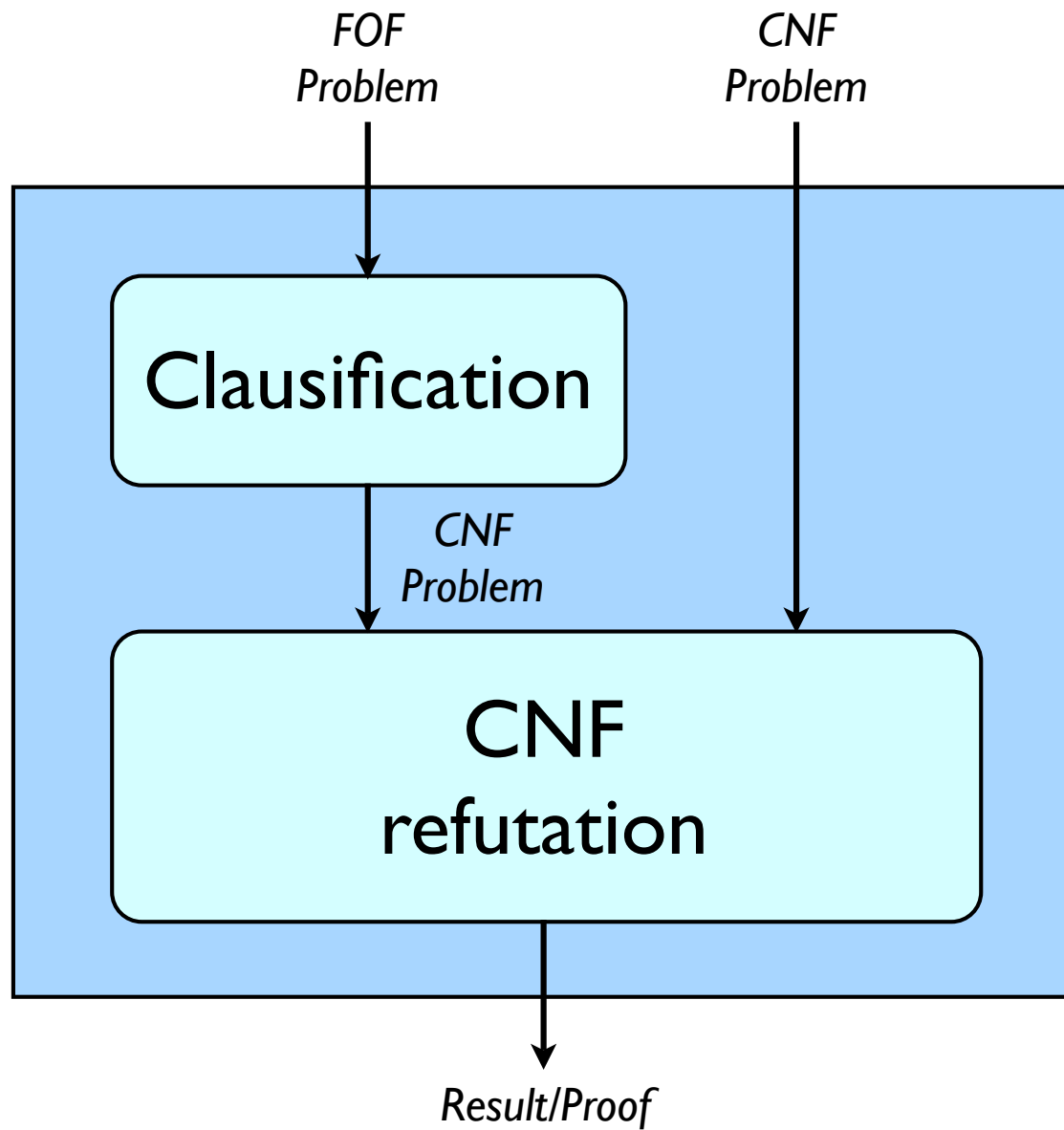
- ▶ Historically: Large variety of syntaxes
  - Prolog-inspired, e.g. LOP (SETHEO, E)
  - By committee, e.g. DFG-Syntax (SPASS)
  - LISP-inspired (SNARK)
  - Home-grown (Otter, Prover-9)
  - TPTP-1/2 syntax (with TPTP2X converter)
  
- ▶ Recently: Quasi-standardization on TPTP-3 syntax [SSCG06, Sut09]
  - Annotated clauses/formulas
  - Can represent problems and proofs
  - Support in Vampire, SPASS, E, E-SETHEO, iProver,



# A First-Order Prover - Bird's Eye Perspective



# A First-Order Prover - Bird's X-Ray Perspective



# Classification



...such that  
 $\{C_1, C_2, \dots, C_3\}$  is **unsatisfiable**  
iff  
 $A \models H$  holds

## Classification



...such that  
 $\{C_1, C_2, \dots, C_3\}$  is **unsatisfiable**  
iff  
 $A \models H$  holds

## Classification



**White Magic:** Standard conjunctive normal form with Skolemization [Lov78] [NW01]  
(read once)

- ▶ Straightforward
- ▶ CNF can explode (and does, occasionally)

**Black Magic:** Miniscoping and definitions [NW01] (Read twice)

- ▶ Smaller CNF, exponential growths can be controlled
- ▶ Better (smaller) terms, less arity in Skolem functions
- ▶ Implemented in **E**

**Forbidden Magic:** Advanced Skolemization [NW01] (Read five times)

- ▶ Implemented in FLOTTER
- ▶ Theoretically superior, but advantage in practice unclear

## Why FOF at all?

```
% All aircraft are either in lower or in upper airspace
fof(low_up_is_exhaustive, axiom,
    (![X]:(lowairspace(X)|uppairspace(X)))).

fof(filter_equiv, conjecture, (
% Naive version: Display aircraft in the Abu Dhabi Approach area in
% lower airspace, display aircraft in the Dubai Approach area in lower
% airspace, display all aircraft in upper airspace, except for
% aircraft in military training region if they are actual military
% aircraft.
    (![X]:(((a_d_app(X) & lowairspace(X))|(dub_app(X) & lowairspace(X))
    |uppairspace(X))&
    (~milregion(X)|~military(X))))
    <=>
% Optimized version: Display all aircraft in either Approach, display
% aircraft in upper airspace, except military aircraft in the military
% training region
    (![X]:((uppairspace(X) | dub_app(X) | a_d_app(X)) &
    (~military(X) | ~milregion(X)))))).
```

# Why FOF at all?

```
cnf(i_0_1,plain,(lowairspace(X1)|uppairspace(X1))).
cnf(i_0_12,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~uppairspace(esk1_0)|~uppairspace(esk2_0))).
cnf(i_0_8,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~uppairspace(esk1_0)|~a_d_app(esk2_0))).
cnf(i_0_10,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~uppairspace(esk1_0)|~dub_app(esk2_0))).
cnf(i_0_13,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~uppairspace(esk2_0))).
cnf(i_0_9,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~a_d_app(esk2_0))).
cnf(i_0_11,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~dub_app(esk2_0))).
cnf(i_0_6,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~uppairspace(esk1_0)|~uppairspace(esk2_0))).
cnf(i_0_2,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~uppairspace(esk1_0)|~a_d_app(esk2_0))).
cnf(i_0_4,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~uppairspace(esk1_0)|~dub_app(esk2_0))).
cnf(i_0_7,negated_conjecture,(military(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~uppairspace(esk2_0))).
cnf(i_0_3,negated_conjecture,(military(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~a_d_app(esk2_0))).
cnf(i_0_5,negated_conjecture,(military(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~dub_app(esk2_0))).
cnf(i_0_36,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
~a_d_app(esk1_0))).
cnf(i_0_24,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
~dub_app(esk1_0))).
cnf(i_0_32,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
~a_d_app(esk2_0))).
cnf(i_0_34,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
~dub_app(esk2_0))).
cnf(i_0_20,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk2_0)|
~dub_app(esk1_0))).
cnf(i_0_22,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~lowairspace(esk1_0)|~dub_app(esk1_0)|
~dub_app(esk2_0))).
cnf(i_0_37,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
~a_d_app(esk1_0))).
cnf(i_0_25,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
~dub_app(esk1_0))).
cnf(i_0_33,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
~a_d_app(esk2_0))).
```

```

cnf(i_0_35,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
~dub_app(esk2_0))).
cnf(i_0_21,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk2_0)|
~dub_app(esk1_0))).
cnf(i_0_23,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~dub_app(esk1_0)|
~dub_app(esk2_0))).
cnf(i_0_30,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
~a_d_app(esk1_0))).
cnf(i_0_18,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
~dub_app(esk1_0))).
cnf(i_0_26,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
~a_d_app(esk2_0))).
cnf(i_0_28,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
~dub_app(esk2_0))).
cnf(i_0_14,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~lowairspace(esk1_0)|~a_d_app(esk2_0)|
~dub_app(esk1_0))).
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~dub_app(esk2_0))).
cnf(i_0_31,negated_conjecture,(military(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
~a_d_app(esk1_0))).
cnf(i_0_19,negated_conjecture,(military(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
~dub_app(esk1_0))).
cnf(i_0_27,negated_conjecture,(military(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
~a_d_app(esk2_0))).
cnf(i_0_29,negated_conjecture,(military(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
~dub_app(esk2_0))).
cnf(i_0_15,negated_conjecture,(military(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk2_0)|
~dub_app(esk1_0))).
cnf(i_0_17,negated_conjecture,(military(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~dub_app(esk1_0)|
~dub_app(esk2_0))).
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dub_app(X1))).
cnf(i_0_39,negated_conjecture,(lowairspace(X2)|uppairspace(X2)|~milregion(X1)|~military(X1))).
cnf(i_0_46,negated_conjecture,(lowairspace(X2)|uppairspace(X2)|uppairspace(X1)|a_d_app(X2)|a_d_app(X1)|

```



```

    dub_app(X1))).
cnf(i_0_45,negated_conjecture,(lowairspace(X2)|uppairspace(X2)|uppairspace(X1)|a_d_app(X1)|
    dub_app(X2)|dub_app(X1))).
cnf(i_0_47,negated_conjecture,(uppairspace(X2)|uppairspace(X1)|a_d_app(X2)|a_d_app(X1)|dub_app(X2)|
    dub_app(X1))).
cnf(i_0_41,negated_conjecture,(lowairspace(X2)|uppairspace(X2)|a_d_app(X2)|~milregion(X1)|~military(X1))).
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cnf(i_0_42,negated_conjecture,(uppairspace(X2)|a_d_app(X2)|dub_app(X2)|~milregion(X1)|~military(X1))).
cnf(i_0_43,negated_conjecture,(uppairspace(X1)|a_d_app(X1)|dub_app(X1)|~milregion(X2)|~military(X2))).
cnf(i_0_38,negated_conjecture,(~milregion(X2)|~milregion(X1)|~military(X2)|~military(X1))).

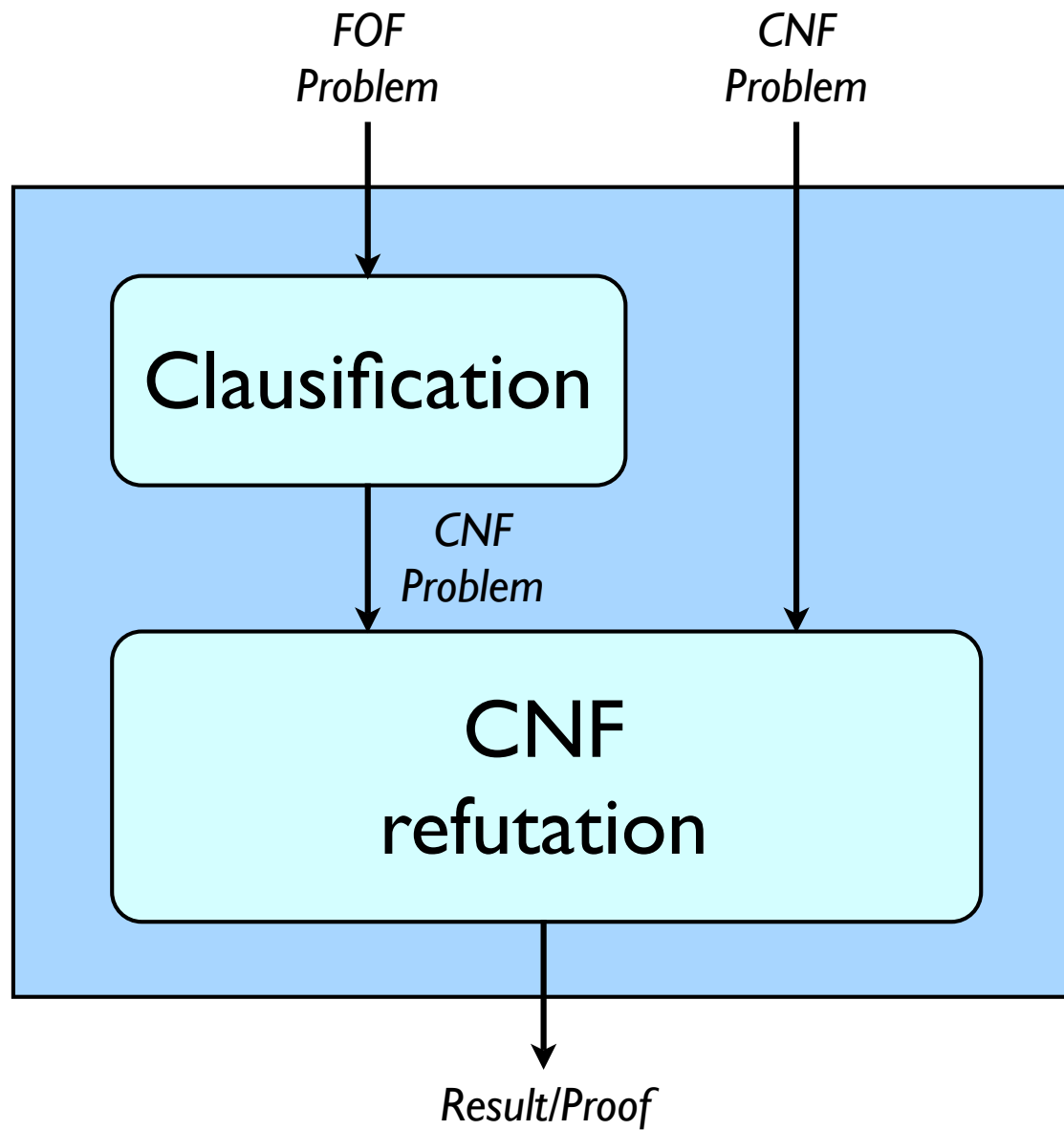
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## Lazy Developer's Clausification



- ▶ iProver (uses E, Vampire)
- ▶ E-SETHEO (uses E, FLOTTER)
- ▶ Fampire (uses FLOTTER)

# A First-Order Prover - Bird's X-Ray Perspective



# CNF Saturation

- ▶ Basic idea: Proof state is a set of clauses  $S$ 
  - Goal: Show unsatisfiability of  $S$
  - Method: Derive **empty clause** via deduction
  - Problem: Proof state explosion
- ▶ Generation: Deduce new clauses
  - Logical core of the calculus
  - Necessary for completeness
  - Lead to explosion is proof state size
    - ⇒ Restrict as much as possible
- ▶ Simplification: Remove or simplify clauses from  $S$ 
  - Critical for acceptable performance
  - Burns most CPU cycles
    - ⇒ Efficient implementation necessary

# Rewriting

▶ Ordered application of equations

- **Replace** equals with equals. . .
- . . . if this decreases term size with respect to given ordering  $>$

$$\frac{s \simeq t \quad u \dot{\simeq} v \vee R}{\frac{s \simeq t \quad u[p \leftarrow \sigma(t)] \dot{\simeq} v \vee R}$$

▶ Conditions:

- $u|_p = \sigma(s)$
- $\sigma(s) > \sigma(t)$
- Some restrictions on rewriting  $>$ -maximal terms in a clause apply

▶ Note: If  $s > t$ , we call  $s \simeq t$  a **rewrite rule**

- Implies  $\sigma(s) > \sigma(t)$ , no ordering check necessary

# Paramodulation/Superposition

- ▶ Superposition: “Lazy conditional speculative rewriting”
  - Conditional: Uses non-unit clauses
    - \* One positive literal is seen as potential rewrite rule
    - \* All other literals are seen as (positive and negative) conditions
  - Lazy: Conditions are not solved, but appended to result
  - Speculative:
    - \* Replaces **potentially** larger terms
    - \* Applies to instances of clauses (generated by unification)
    - \* Original clauses remain (generating inference)

$$\frac{s \simeq t \vee S \quad u \simeq v \vee R}{\sigma(u[p \leftarrow t] \simeq v \vee S \vee R)}$$

- ▶ Conditions:
  - $\sigma = mgu(u|_p, s)$  and  $u|_p$  is not a variable
  - $\sigma(s) \not\prec \sigma(t)$  and  $\sigma(u) \not\prec \sigma(v)$
  - $\sigma(s \simeq t)$  is  $>$ -maximal in  $\sigma(s \simeq t \vee S)$  (and no negative literal is **selected**)
  - $\sigma(u \simeq v)$  is maximal (and no negative literal is selected) or selected

# Subsumption

- ▶ Idea: Only keep the most general clauses
  - If one clause is **subsumed** by another, discard it

$$\frac{C \quad \sigma(C) \vee R}{C}$$

- ▶ Examples:
  - $p(X)$  subsumes  $p(a) \vee q(f(X), a)$  ( $\sigma = \{X \leftarrow a\}$ )
  - $p(X) \vee p(Y)$  does not multi-set-subsume  $p(a) \vee q(f(X), a)$
  - $q(X, Y) \vee q(X, a)$  subsumes  $q(a, a) \vee q(a, b)$
- ▶ Subsumption is hard (NP-complete)
  - $n!$  permutations in non-equational clause with  $n$  literals
  - $n!2^n$  permutations in equational clause with  $n$  literals

# Term Orderings

- ▶ Superposition is instantiated with a ground-completable simplification ordering  $>$  on terms
  - $>$  is Noetherian
  - $>$  is compatible with term structure:  $t_1 > t_2$  implies  $s[t_1]_p > s[t_2]_p$
  - $>$  is compatible with substitutions:  $t_1 > t_2$  implies  $\sigma(t_1) > \sigma(t_2)$
  - $>$  has the subterm-property:  $s > s|_p$
  - In practice: LPO, KBO, RPO
- ▶ Ordering evaluation is one of the major costs in superposition-based theorem proving
- ▶ Efficient implementation of orderings: [Löc06, LÖ6]



# Generalized Redundancy Elimination

- ▶ A clause is redundant in  $S$ , if all its ground instances are implied by  $>$  smaller ground instances of other clauses in  $S$ 
  - May require addition of smaller implied clauses!
- ▶ Examples:
  - Rewriting (rewritten clause added!)
  - Tautology deletion (implied by empty clause)
  - Redundant literal elimination:  $l \vee l \vee R$  replaced by  $l \vee R$
  - False literal elimination:  $s \neq s \vee R$  replaced by  $R$
- ▶ Literature:
  - Theoretical results: [BG94, BG98, NR01]
  - Some important refinements used in E: [Sch02, Sch04b, RV01, Sch09]

# The Basic Given-Clause Algorithm

- ▶ Completeness requires consideration of all possible persistent clause combinations for generating inferences
  - For superposition: All 2-clause combinations
  - Other inferences: Typically a single clause
- ▶ Given-clause algorithm replaces complex bookkeeping with simple invariant:
  - Proofstate  $S = P \cup U$ ,  $P$  initially empty
  - All inferences between clauses in  $P$  have been performed
- ▶ The algorithm:

```
while  $U \neq \{\}$   
   $g = \text{delete\_best}(U)$   
  if  $g == \square$   
    SUCCESS, Proof found  
   $P = P \cup \{g\}$   
   $U = U \cup \text{generate}(g, P)$   
SUCCESS, original  $U$  is satisfiable
```

# DISCOUNT Loop

- ▶ Aim: Integrate simplification into given clause algorithm
- ▶ The algorithm (as implemented in E):

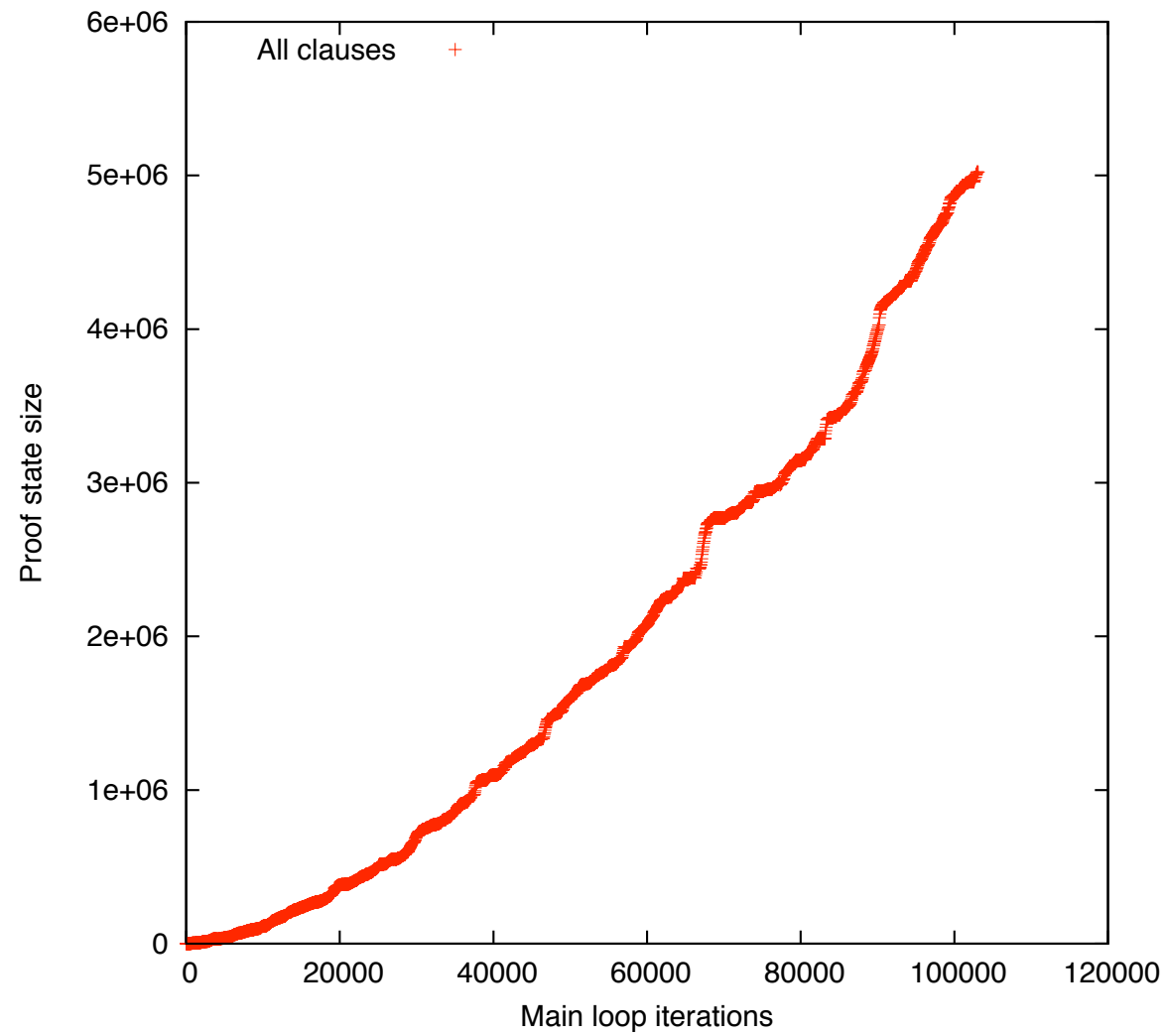
```
while  $U \neq \{\}$ 
   $g = \text{delete\_best}(U)$ 
   $g = \text{simplify}(g, P)$ 
  if  $g == \square$ 
    SUCCESS, Proof found
  if  $g$  is not redundant w.r.t.  $P$ 
     $T = \{c \in P \mid c \text{ redundant or simplifiable w.r.t. } g\}$ 
     $P = (P \setminus T) \cup \{g\}$ 
     $T = T \cup \text{generate}(g, P)$ 
    foreach  $c \in T$ 
       $c = \text{cheap\_simplify}(c, P)$ 
      if  $c$  is not trivial
         $U = U \cup \{c\}$ 
  SUCCESS, original  $U$  is satisfiable
```

# What is so hard about this?

## What is so hard about this?

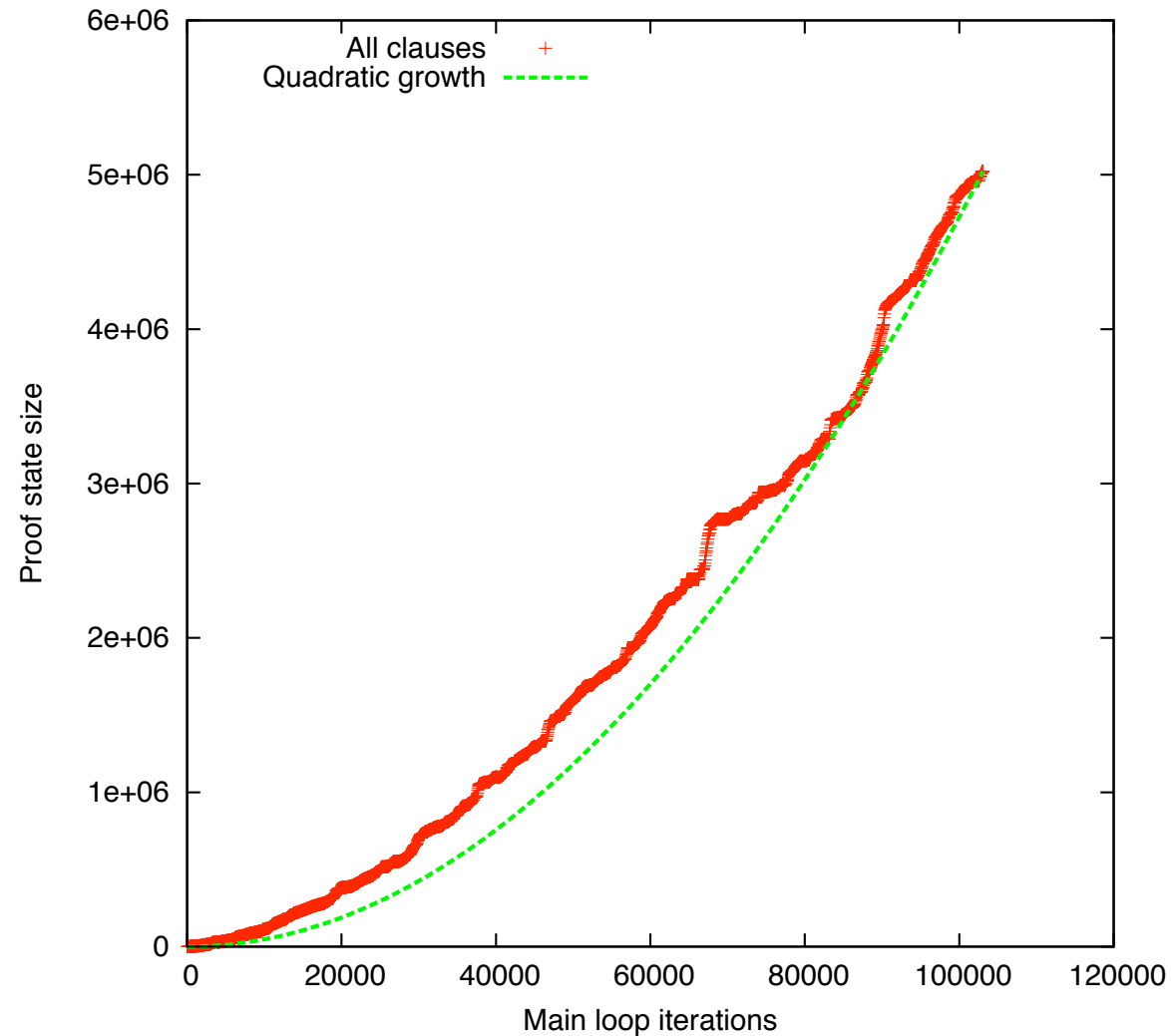
- ▶ Data from **simple** TPTP example NUM030-1+rm\_eq\_rstfp.lop (solved by E in 30 seconds on ancient Apple Powerbook):
  - Initial clauses: 160
  - Processed clauses: 16,322
  - Generated clauses: 204,436
  - Paramodulations: 204,395
  - Current number of processed clauses: 1,885
  - Current number of unprocessed clauses: 94,442
  - Number of terms: 5,628,929
- ▶ Hard problems run for days!
  - Millions of clauses generated (and stored)
  - Many millions of terms stored and rewritten
  - Each rewrite attempt must consider many ( $\gg 10000$ ) rules
  - Subsumption must test many ( $\gg 10000$ ) candidates for each subsumption attempt
  - Heuristic must find best clause out of millions

# Proof State Development



Proof state behavior for ring theory example RNG043-2 (Default Mode)

# Proof State Development



Proof state behavior for ring theory example RNG043-2 (Default Mode)

- ▶ Growth is roughly quadratic in the number of processed clauses

# Literature on Proof Procedures

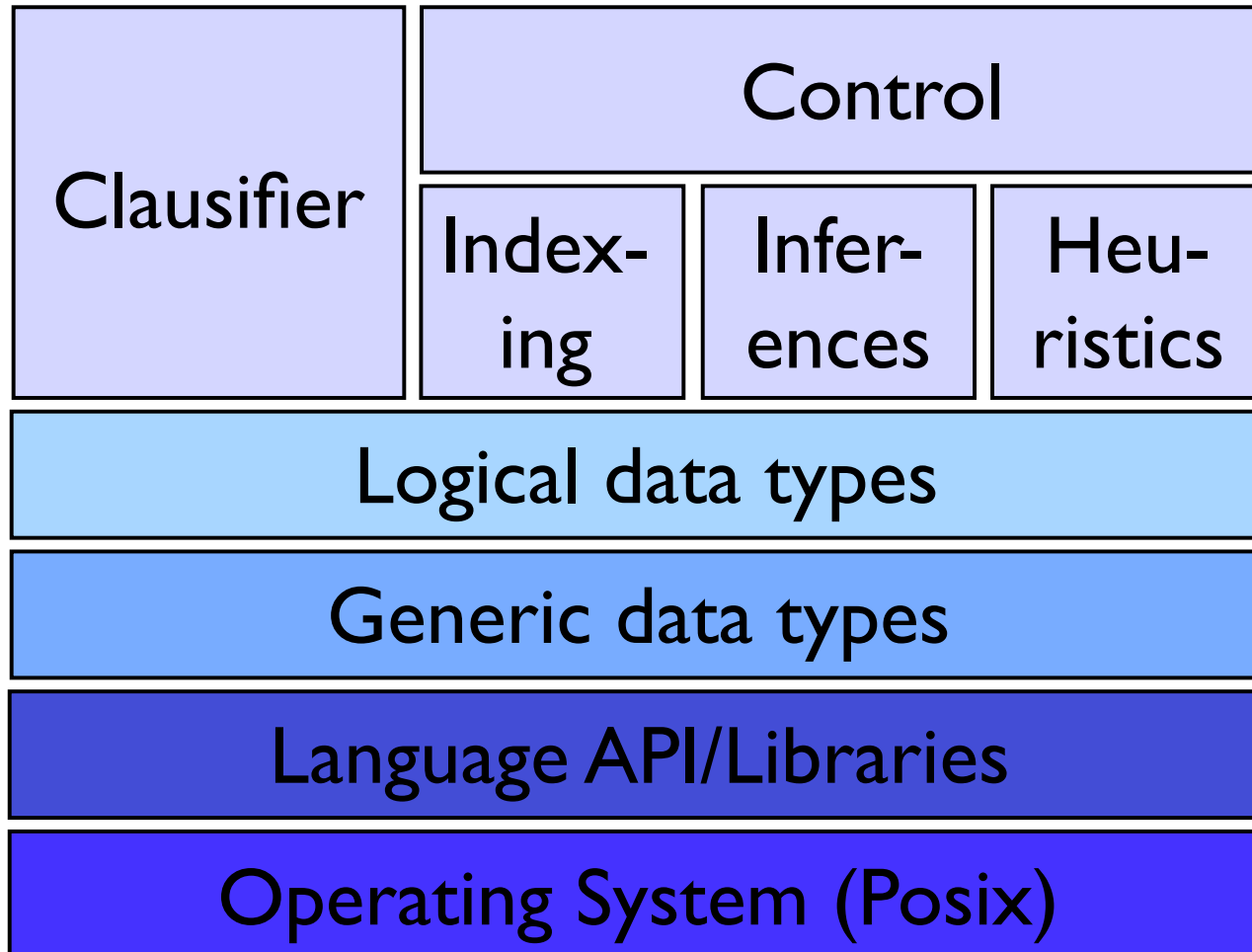
- ▶ New Waldmeister Loop: [GHLS03]
- ▶ Comparisons: [RV03]
- ▶ Best discussion of E Loop: [Sch02]



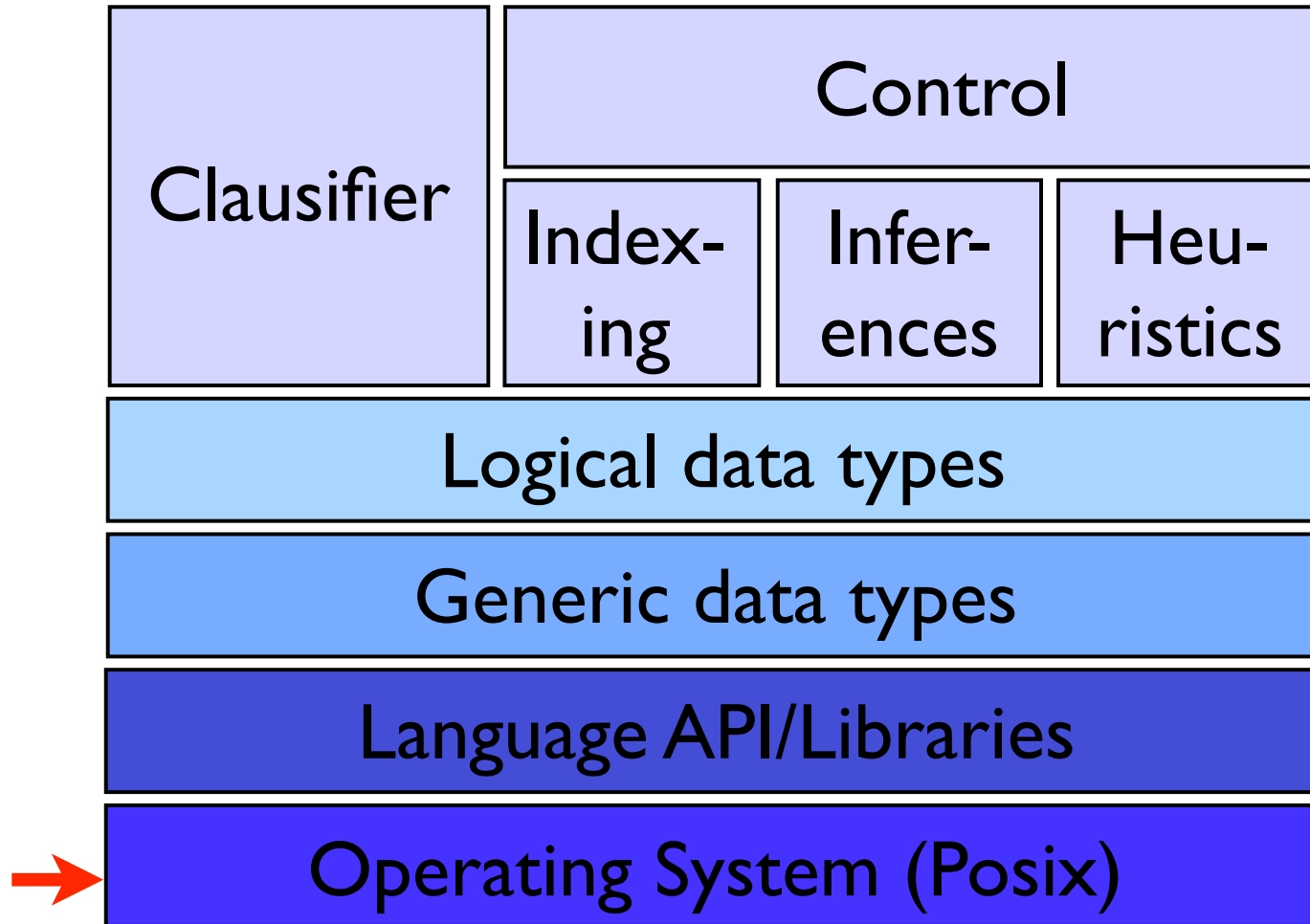
## Exercise: Installing and Running E

- ▶ Goto `http://www.eprover.org`
- ▶ Find the download section
- ▶ Find and read the README
- ▶ Download the source tarball
- ▶ Following the README, build the system in a local user directory
- ▶ Run the prover on one of the included examples to demonstrates that it works.

# Layered Architecture



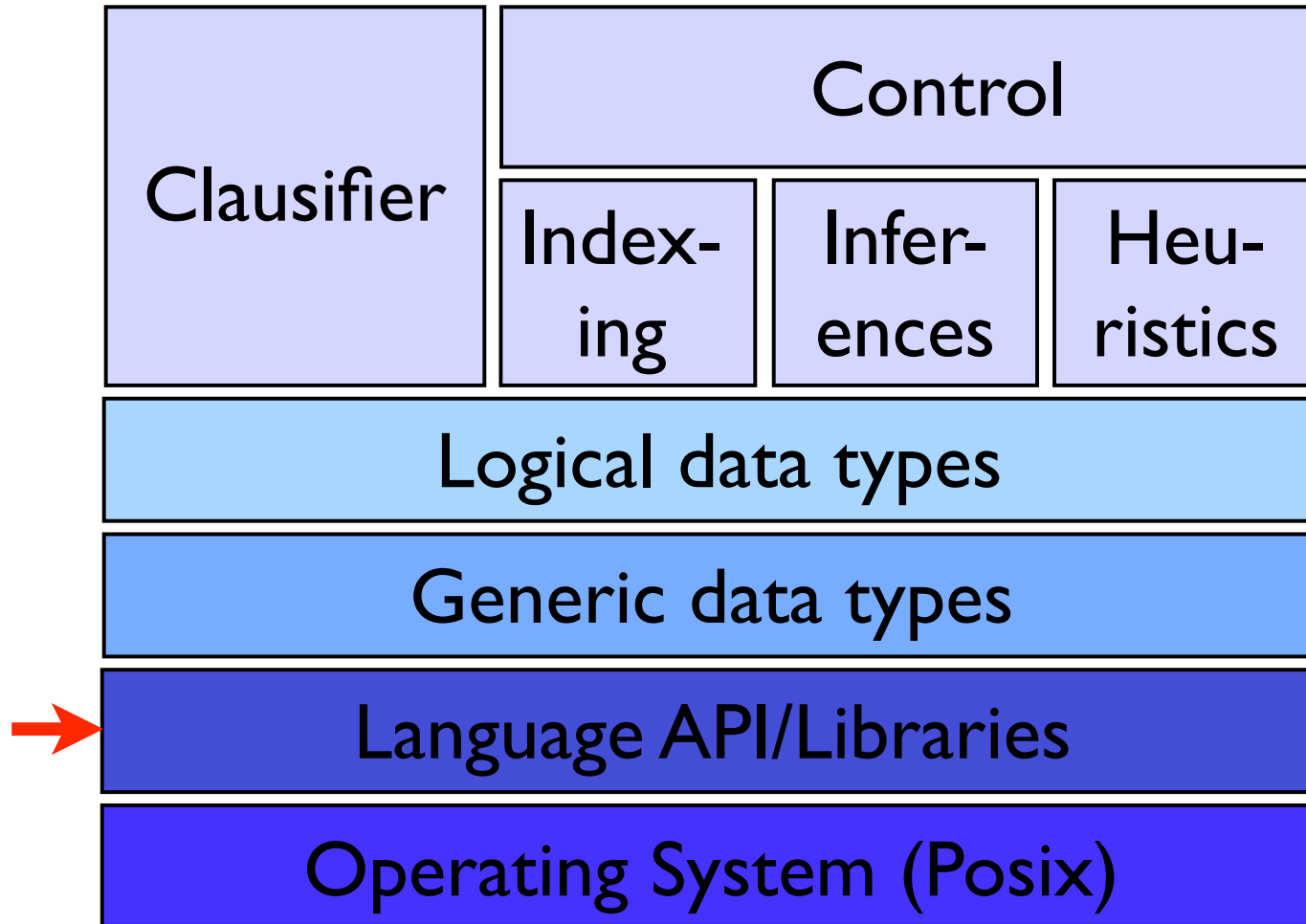
# Layered Architecture



# Operating System

- ▶ Pick a UNIX variant
  - Widely used
  - Free
  - Stable
  - Much better support for remote tests and automation
  - Everybody else uses it ;-)
  
- ▶ Aim for portability
  - Theorem provers have minimal requirements
  - Text input/output
  - POSIX is sufficient

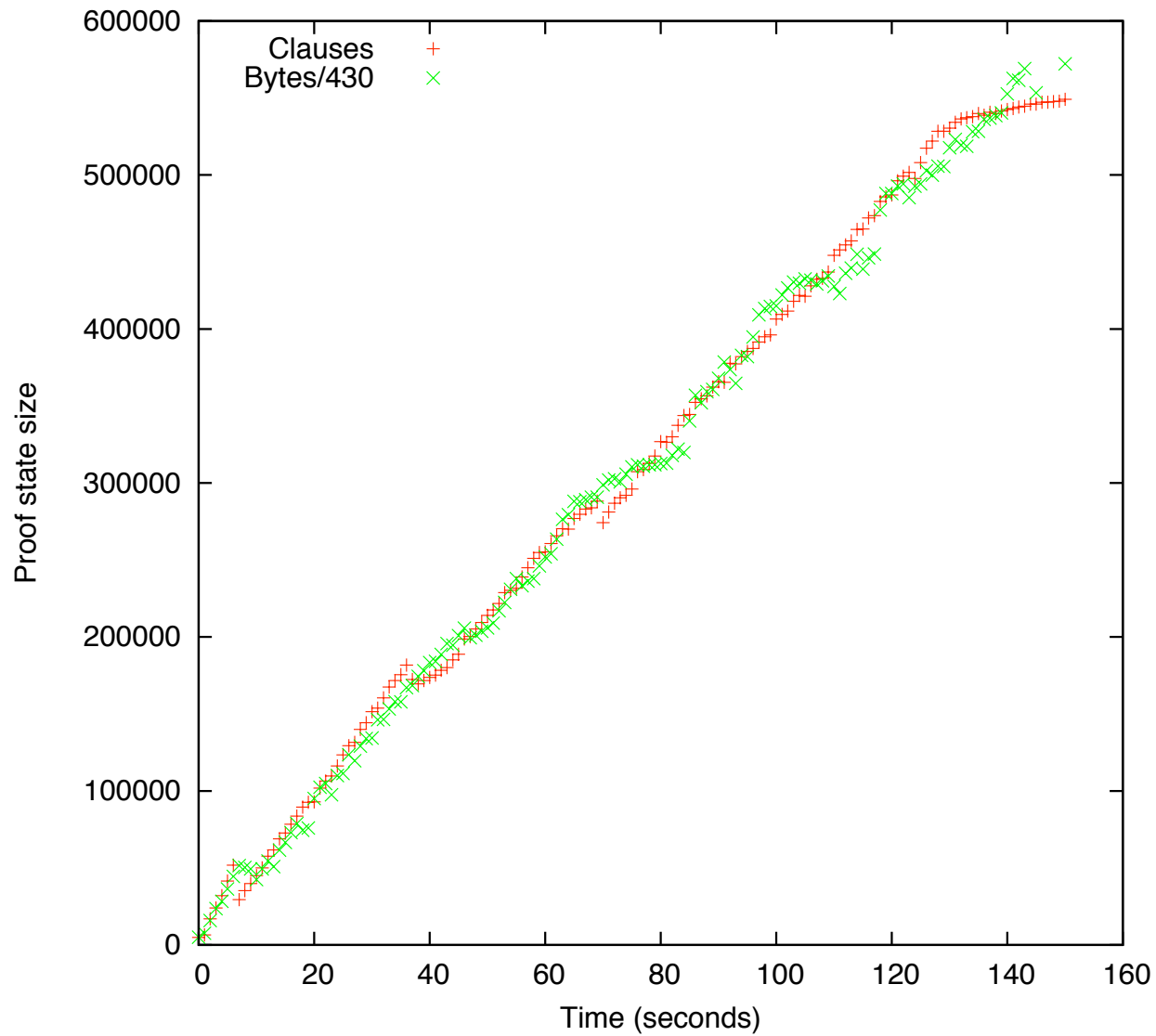
# Layered Architecture



# Language API/Libraries

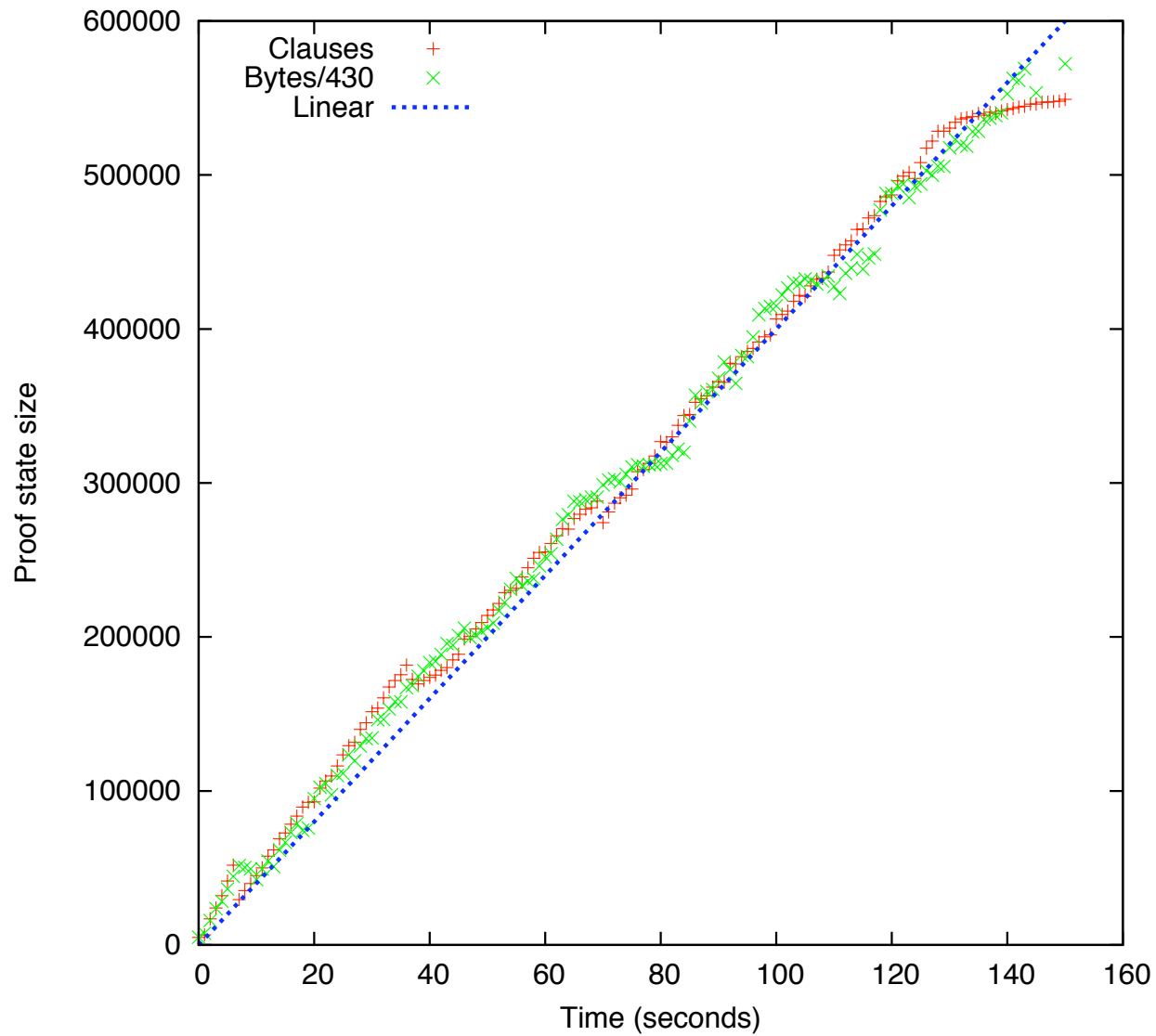
- ▶ Pick your language
- ▶ High-level/functional or declarative languages come with rich datatypes and libraries
  - Can cover "Generic data types"
  - Can even cover 90% of "Logical data types"
- ▶ C offers nearly full control
  - Much better for low-level performance
  - . . . if you can make it happen!

# Memory Consumption



- ▶ Proof state behavior for number theory example NUM030-1 (880 MHz SunFire)

# Memory Consumption



- ▶ Proof state behavior for number theory example NUM030-1 (880 MHz SunFire)

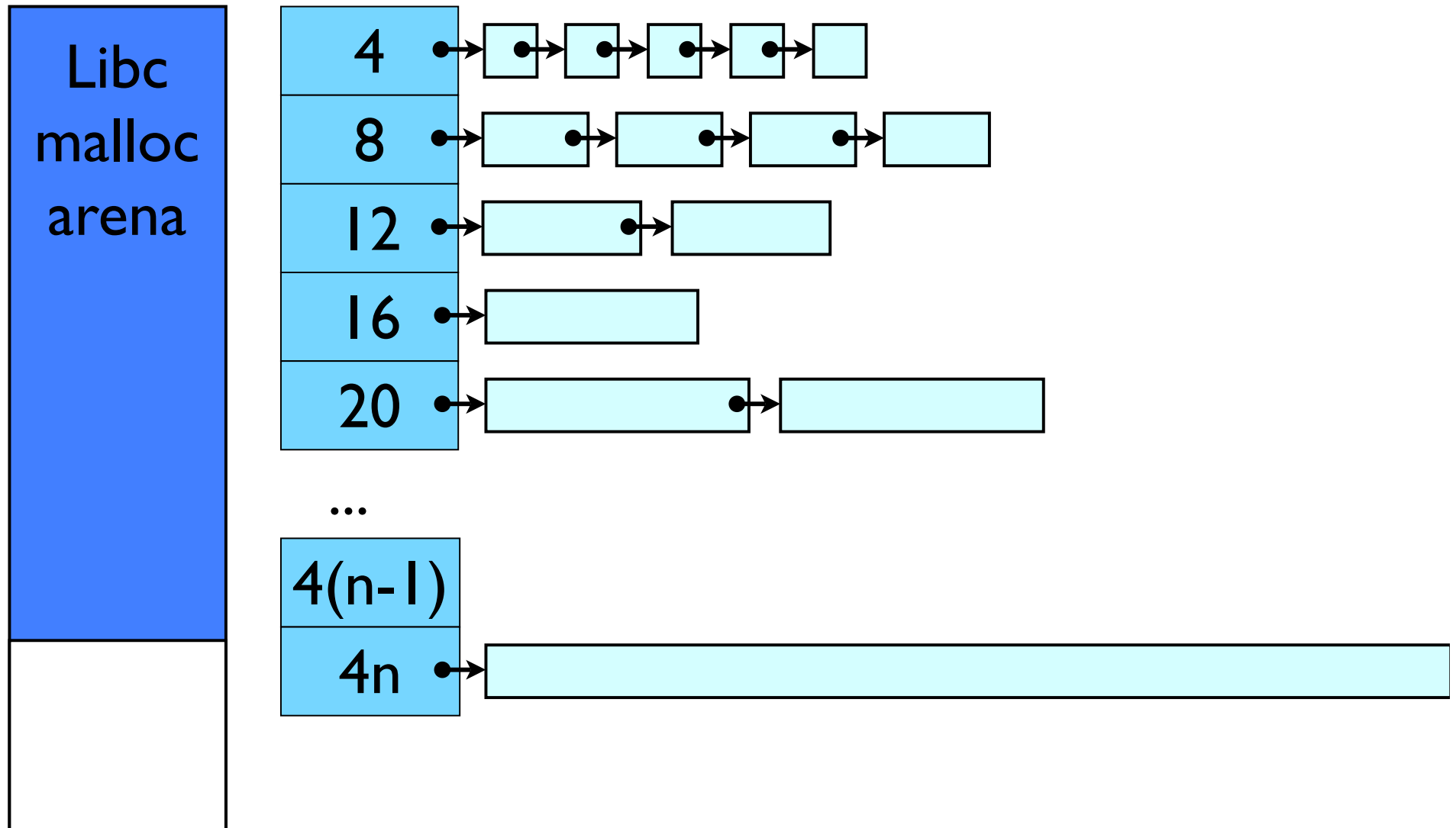


# Memory Management

- ▶ Nearly all memory in a saturating prover is taken up by very few data types
  - Terms
  - Literals
  - Clauses
  - Clause evaluations
  - (Indices)
- ▶ These data types are frequently created and destroyed
  - Prime target for freelist based memory management
  - Backed directly by system malloc()
  - Allocating and chopping up large blocks does not pay off!
- ▶ Result:
  - Allocating temporary data structures is  $O(1)$
  - Overhead is very small
  - Speedup 20%-50% depending on OS/processor/libC version

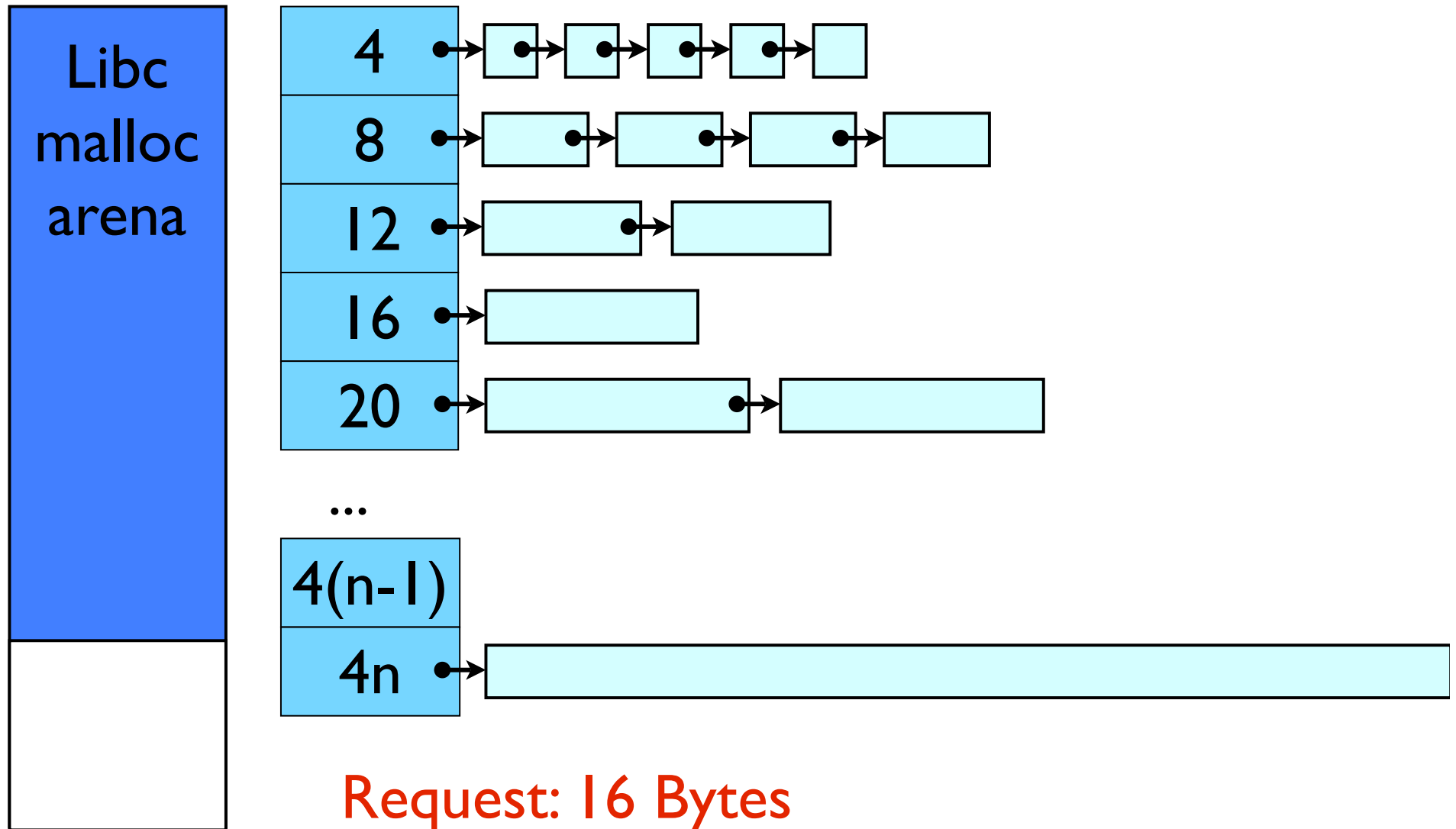
# Memory Management illustrated

Anchors      Free lists

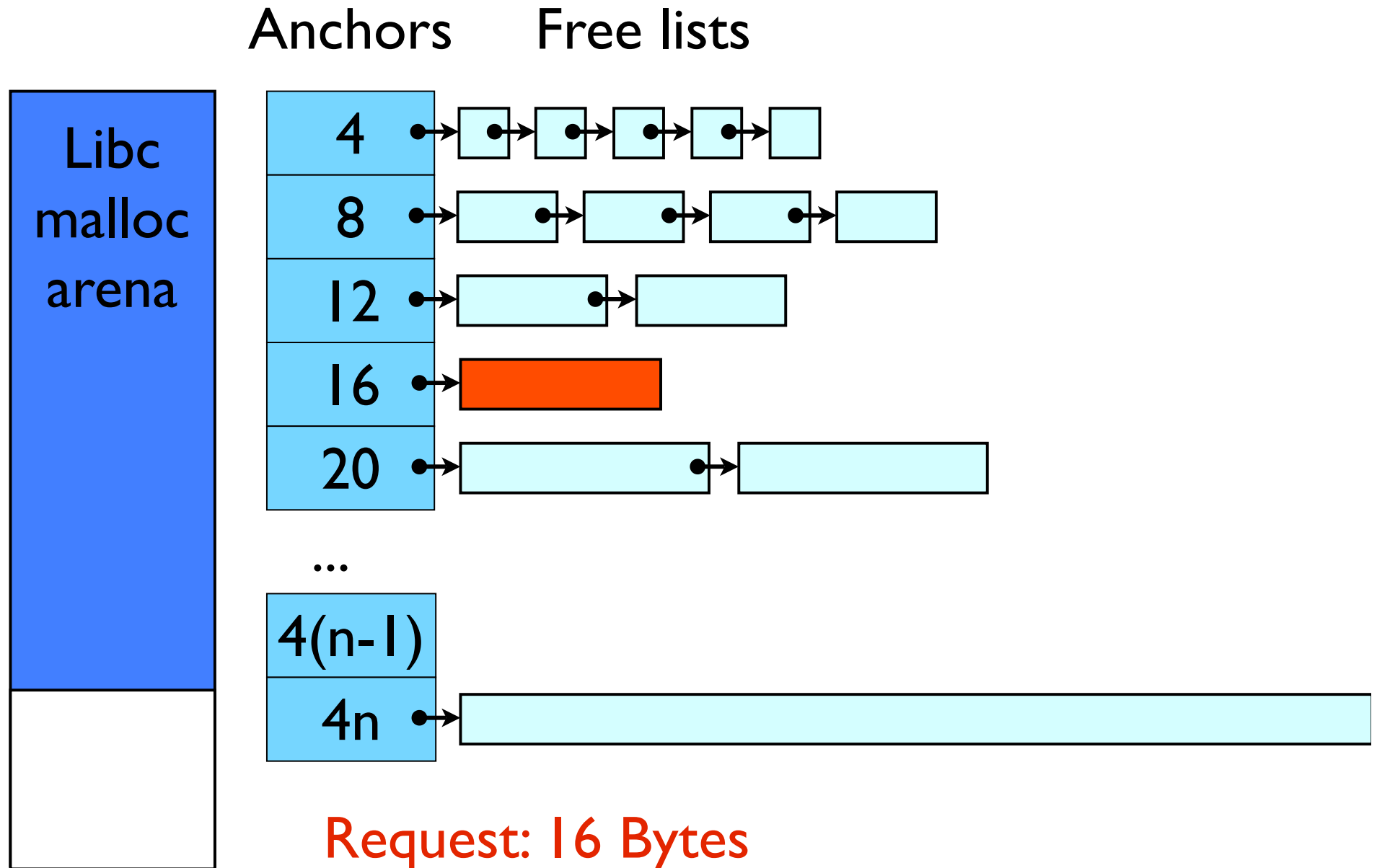


# Memory Management illustrated

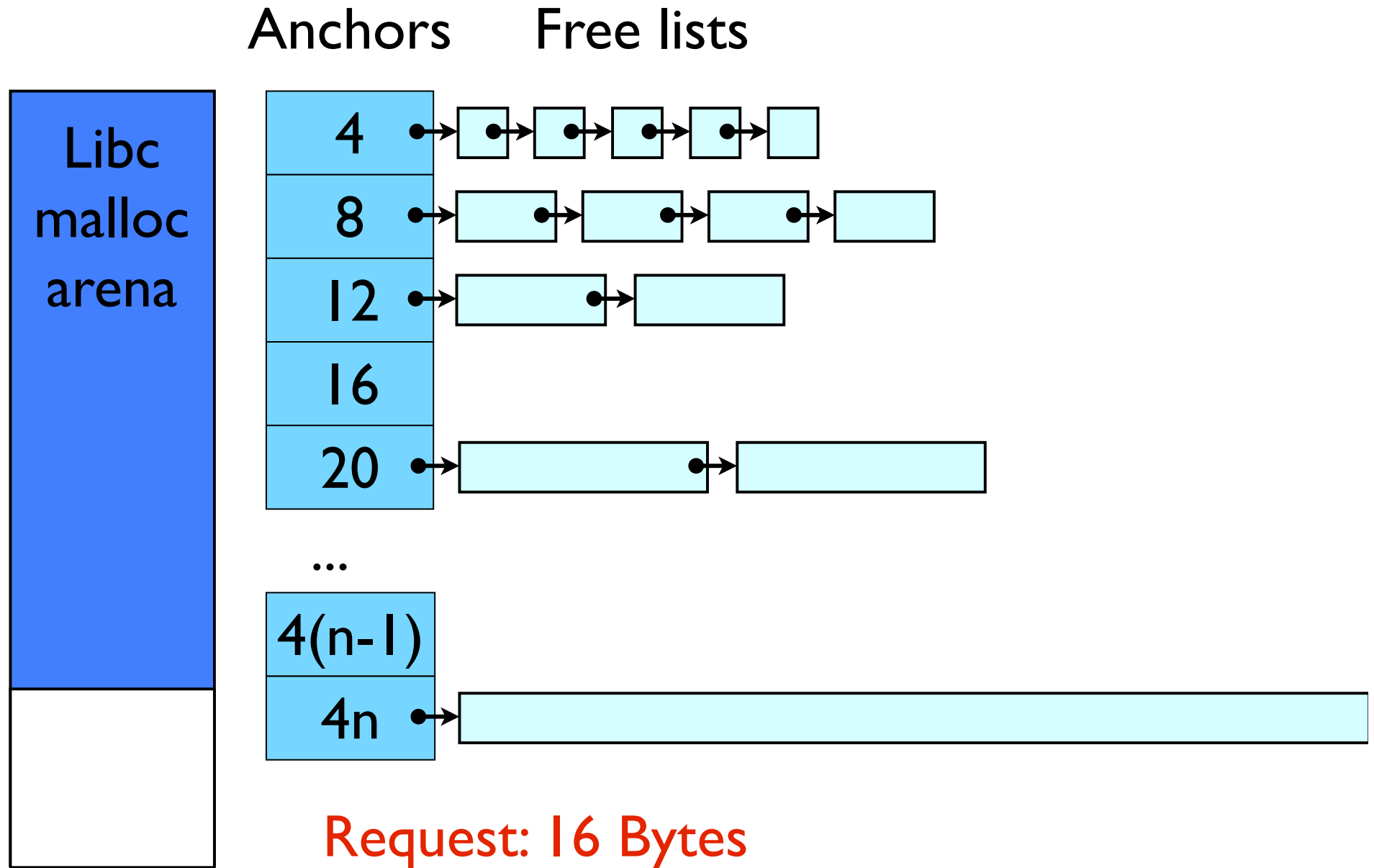
Anchors      Free lists



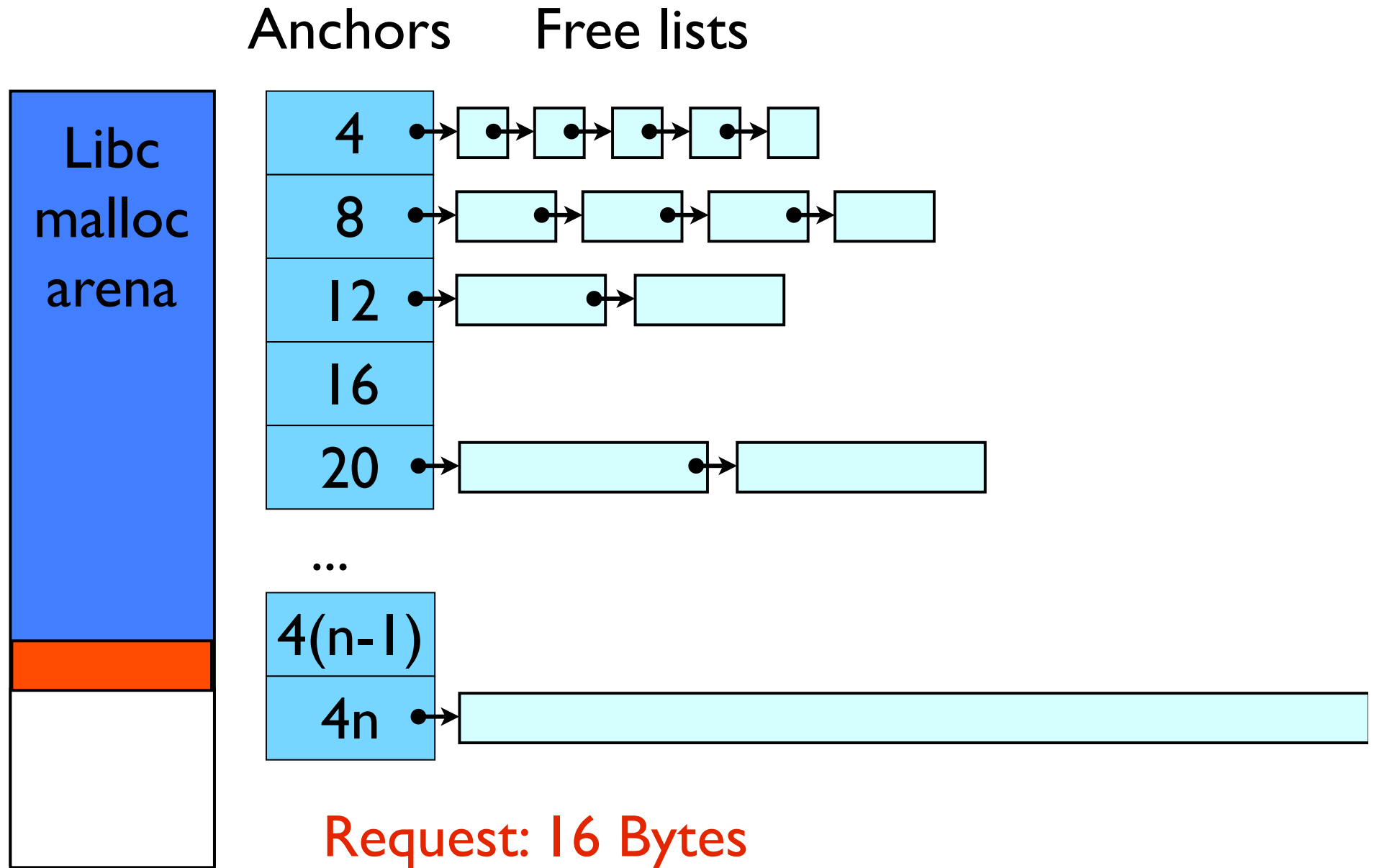
# Memory Management illustrated



# Memory Management illustrated

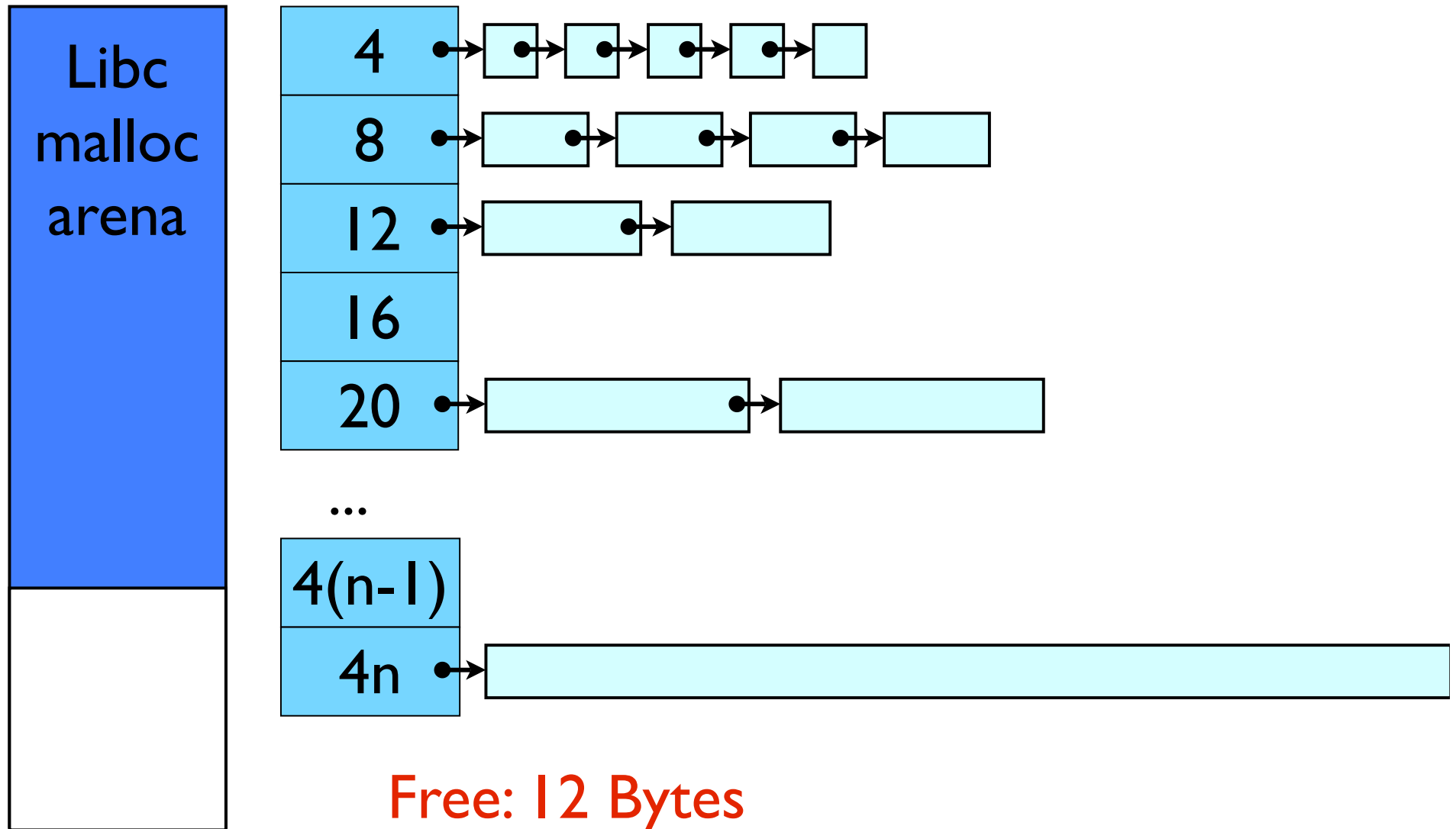


# Memory Management illustrated



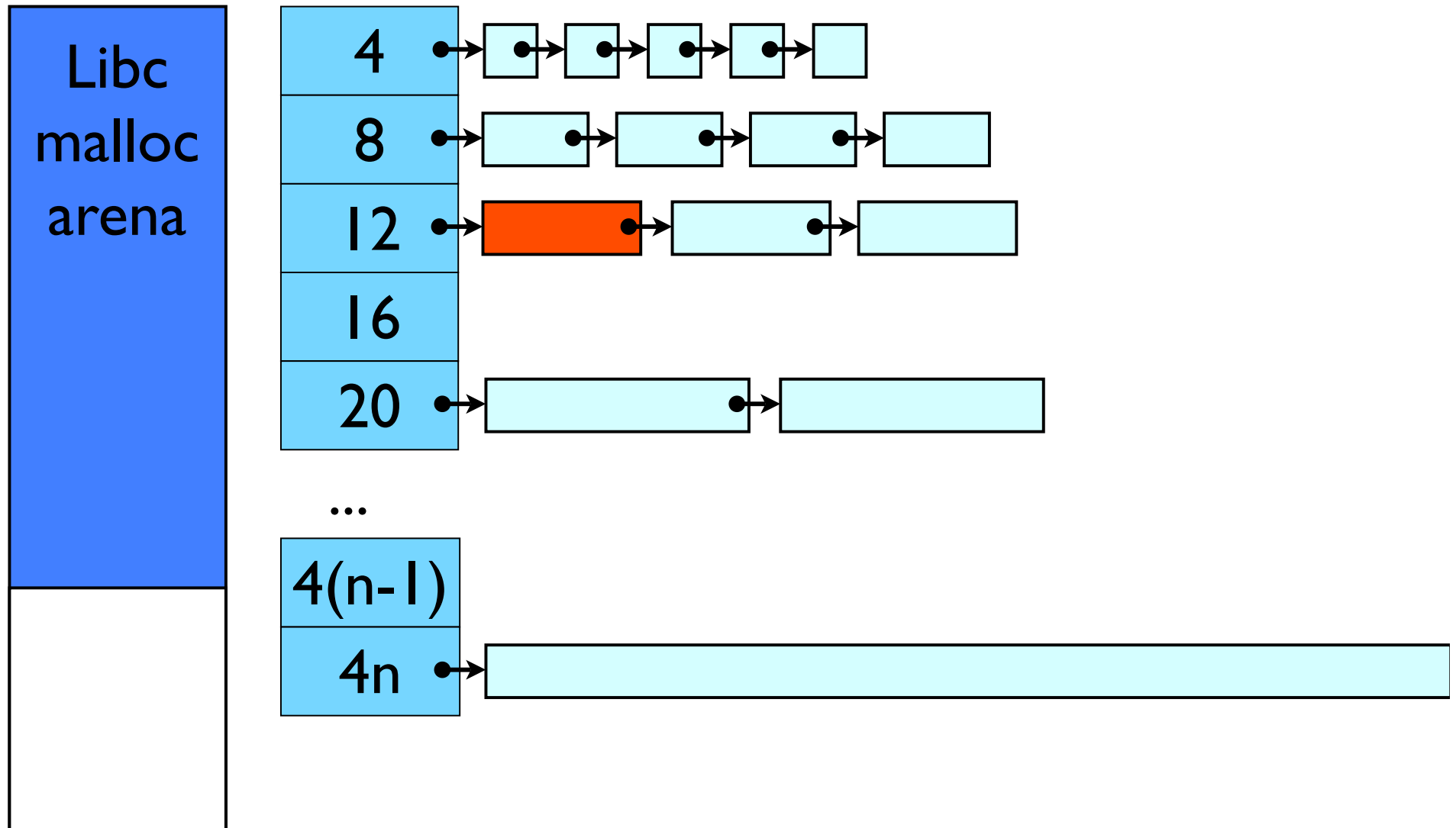
# Memory Management illustrated

Anchors      Free lists



# Memory Management illustrated

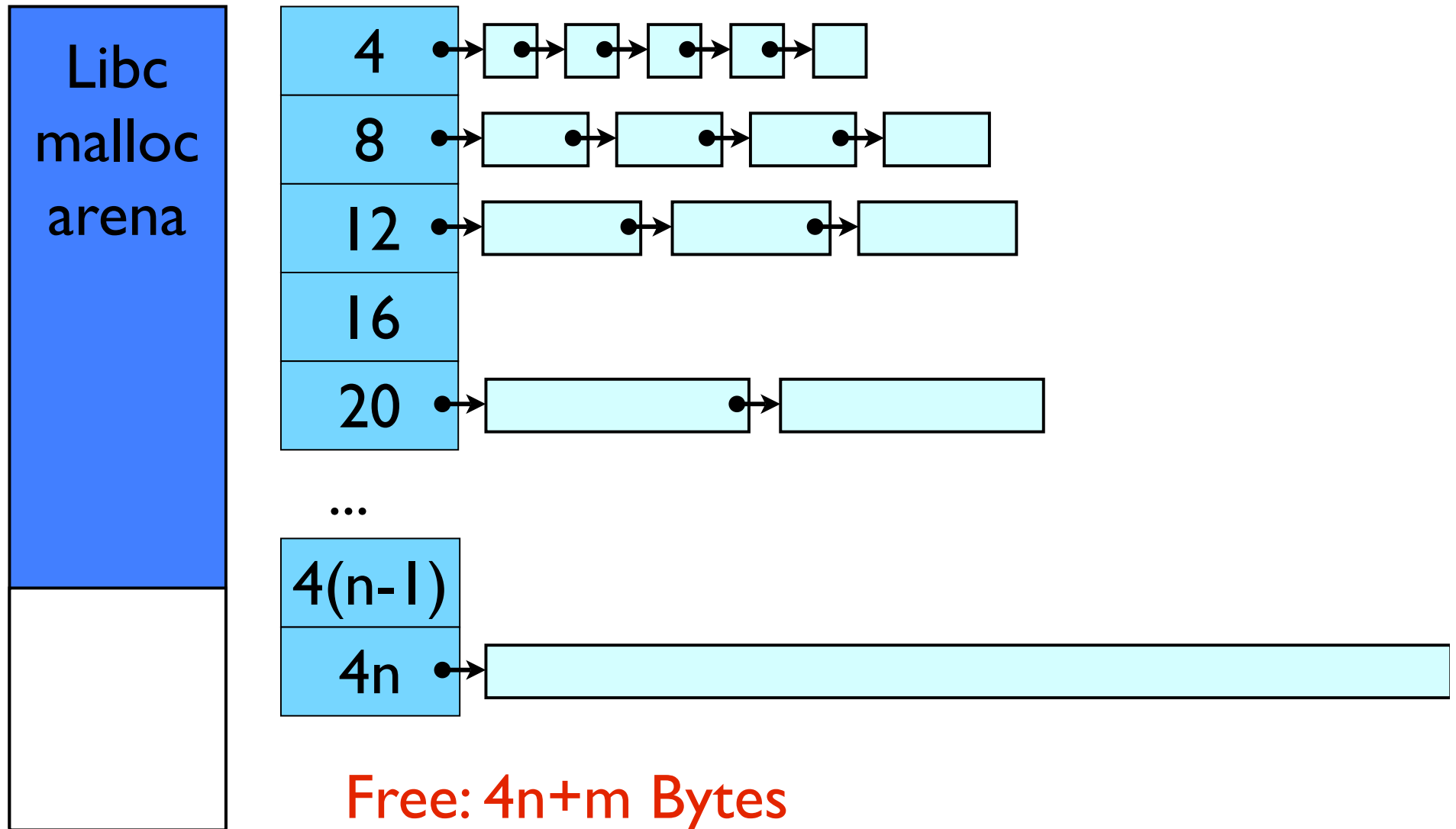
Anchors      Free lists





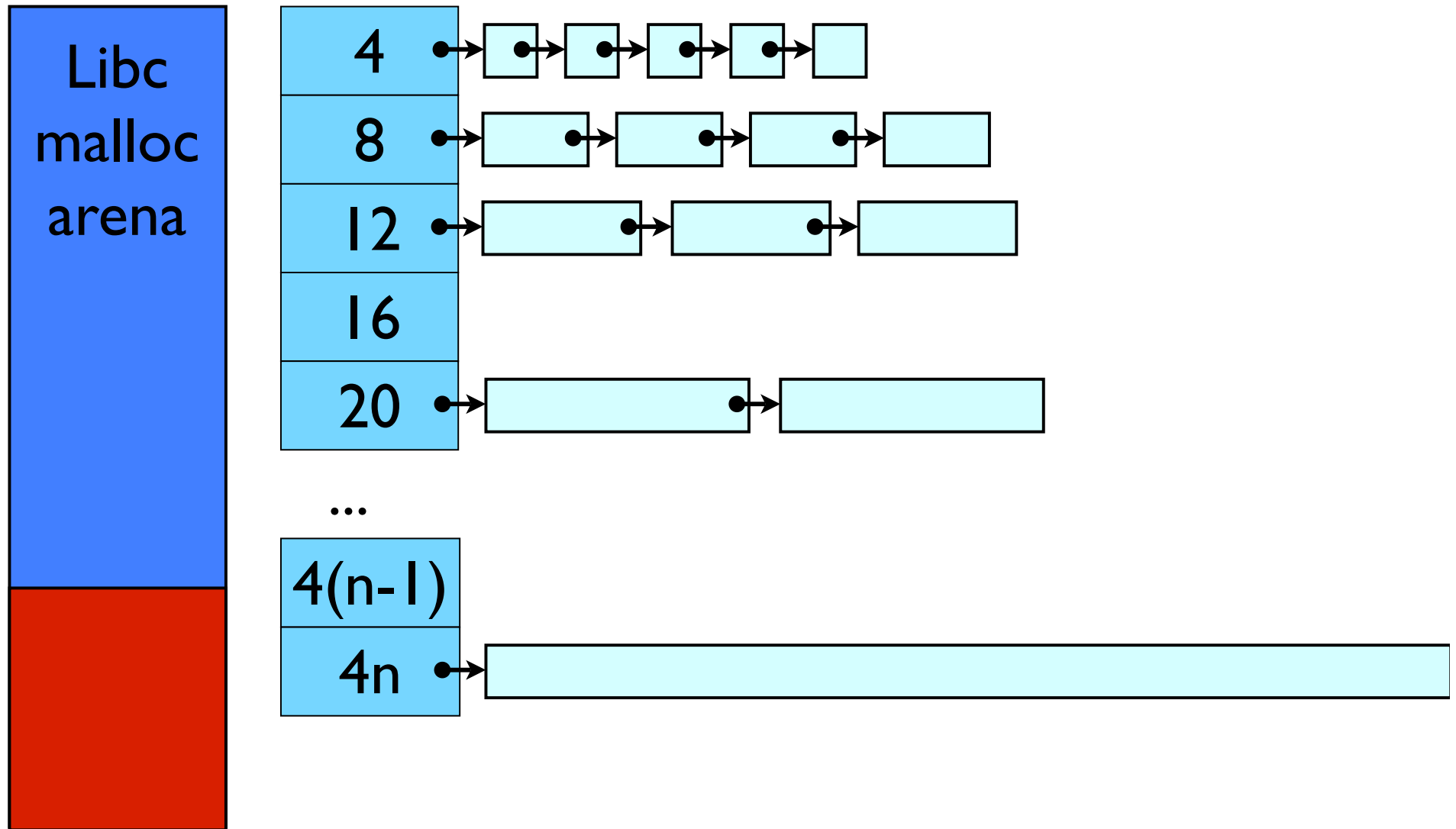
# Memory Management illustrated

Anchors      Free lists



# Memory Management illustrated

Anchors      Free lists



## Exercise: Influence of Memory Management

- ▶ E can be build with 2 different workin memory management schemes
  - Vanilla libC malloc()
    - \* Add compiler option `-DUSE_SYSTEM_MEM` in `E/Makefile.vars`
  - Freelists backed by malloc() (see above)
    - \* Default version
  
- ▶ Compare the performance yourself:
  - Run default E a couple of times with output disabled
  - `eprover -s --resources-info LUSK6ext.lop`
  - Take note of the reported times
  - Enable use of system malloc(), then `make rebuild`
  - Rerun the tests and compare the times

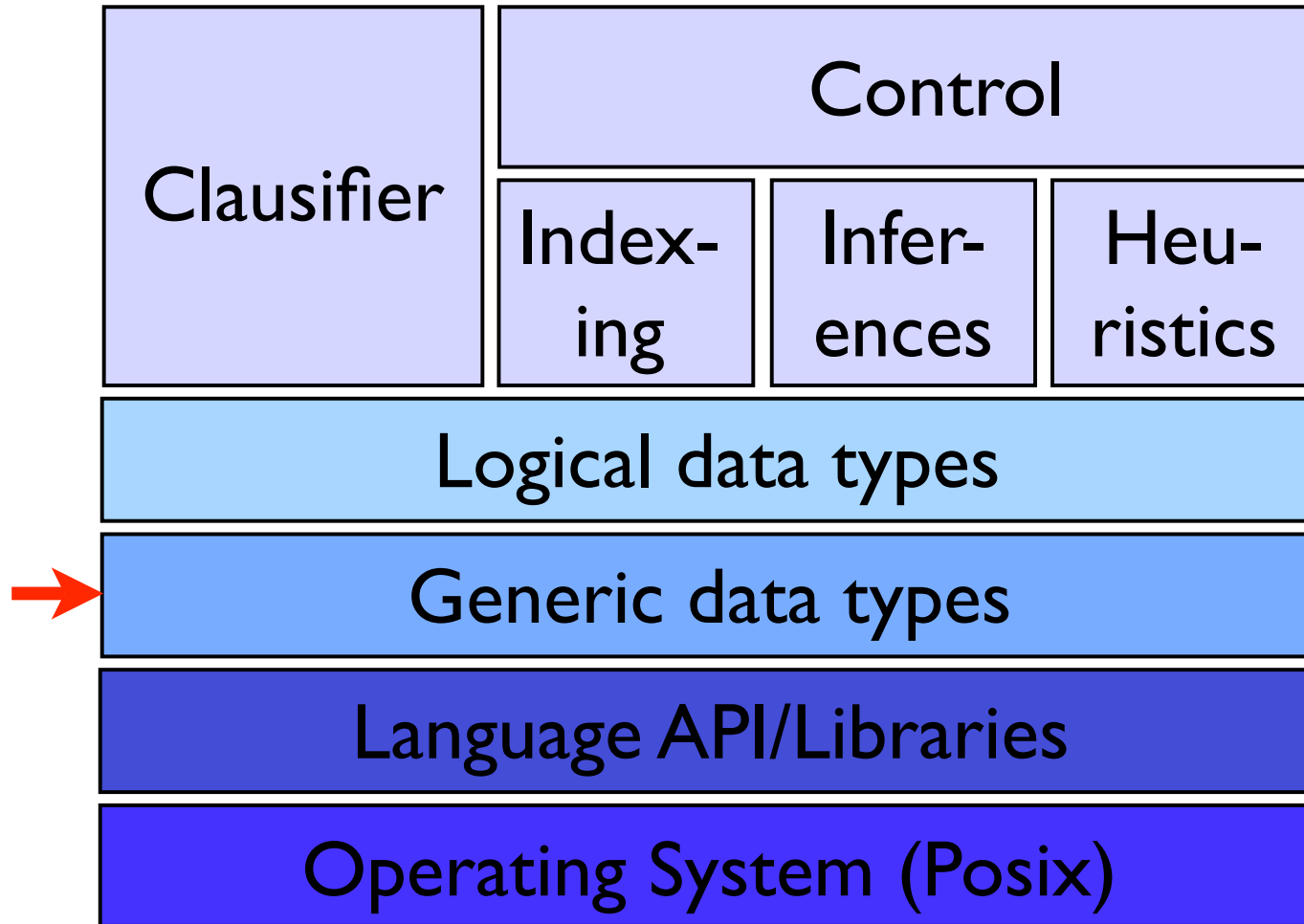
## Makefile.vars

...

```
BUILDFLAGS = -DPRINT_SOMEERRORS_STDOUT \  
             -DMEMORY_RESERVE_PARANOID \  
             -DPRINT_TSTP_STATUS \  
-DSTACK_SIZE=32768 \  
             -DUSE_SYSTEM_MEM \  
             # -DFULL_MEM_STATS\  
             # -DPRINT_RW_STATE # -DMEASURE_EXPENSIVE
```

...

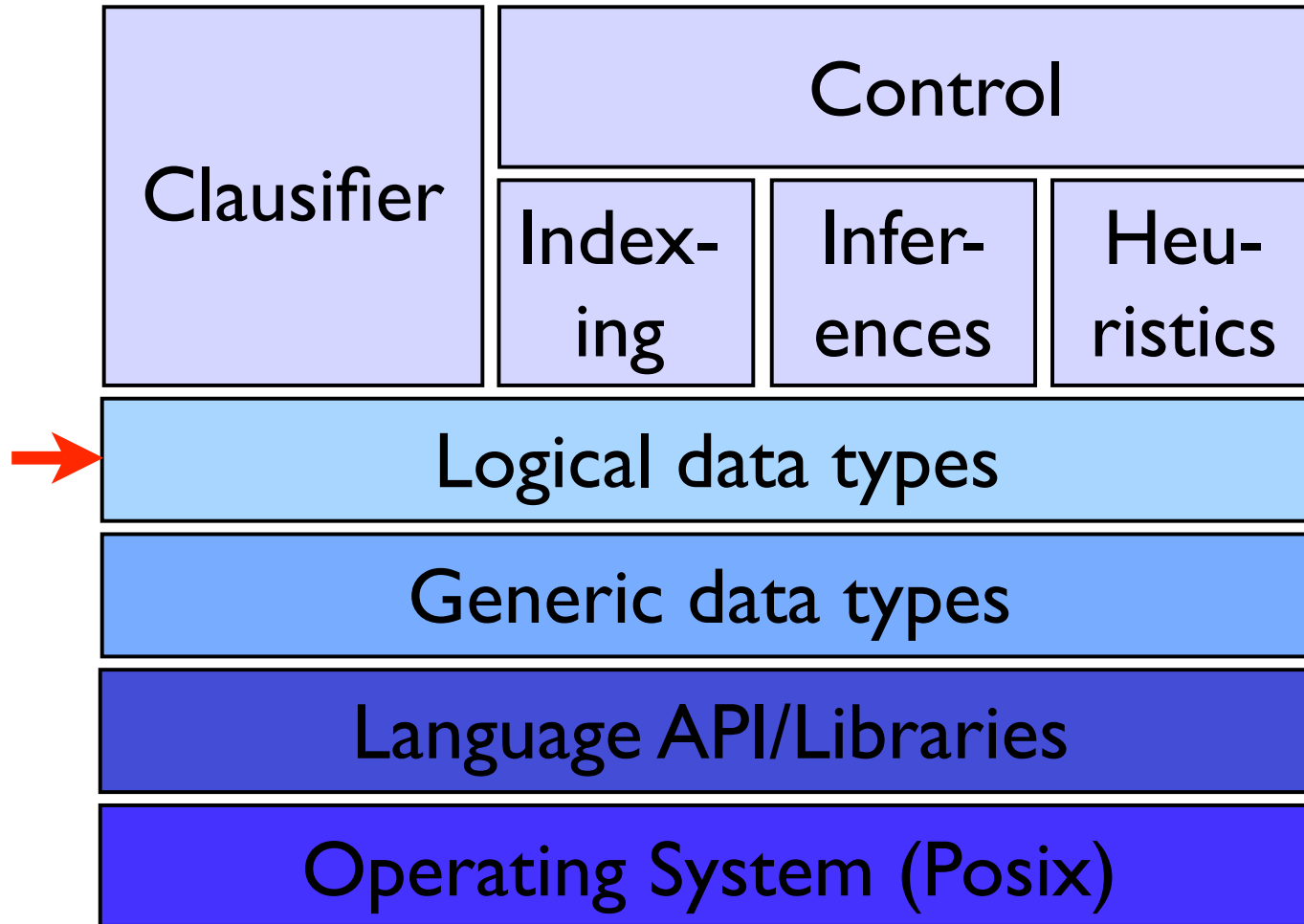
# Layered Architecture



# Generic Data types

- ▶ (Dynamic) Stacks
- ▶ (Dynamic) Arrays
- ▶ Hashes
- ▶ Singly linked lists
- ▶ Doubly linked lists
- ▶ Tries
- ▶ Splay trees [ST85]
- ▶ Skip lists [Pug90]

# Layered Architecture



# First-Order Terms

- ▶ Terms are words over the alphabet  $F \cup V \cup \{ ' ( , ' ) ' , ' , ' \}$ , where. . .
- ▶ Variables:  $V = \{ X, Y, Z, X_1, \dots \}$
- ▶ Function symbols:  $F = \{ f/2, g/1, a/0, b/0, \dots \}$
- ▶ Definition of terms:
  - $X \in V$  is a term
  - $f/n \in F, t_1, \dots, t_n$  are terms  $\rightsquigarrow f(t_1, \dots, t_n)$  is a term
  - Nothing else is a term

Terms are by far the most frequent objects in a typical proof state!

$\rightsquigarrow$  **Term representation is critical!**



# Representing Function Symbols and Variables

- ▶ Naive: Representing function symbols as strings: "f", "g", "add"
  - May be ok for  $f$ ,  $g$ ,  $add$
  - Users write  $unordered\_pair$ ,  $universal\_class$ , ...
- ▶ Solution: Signature table
  - Map each function symbol to unique small positive integer
  - Represent function symbol by this integer
  - Maintain table with meta-information for function symbols indexed by assigned code
- ▶ Handling variables:
  - Rename variables to  $\{X_1, X_2, \dots\}$
  - Represent  $X_i$  by  $-i$
  - Disjoint from function symbol codes!

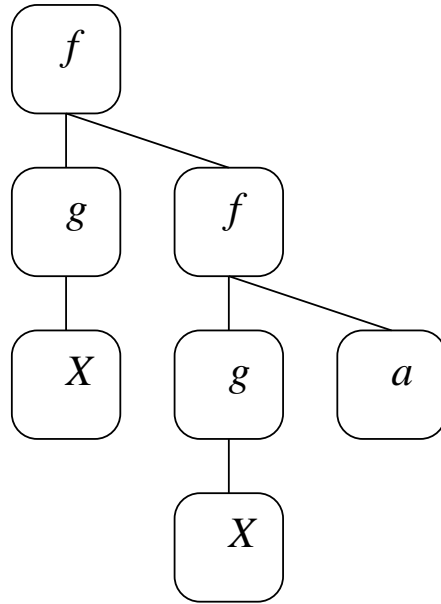
From now on, assume this always done!

# Representing Terms

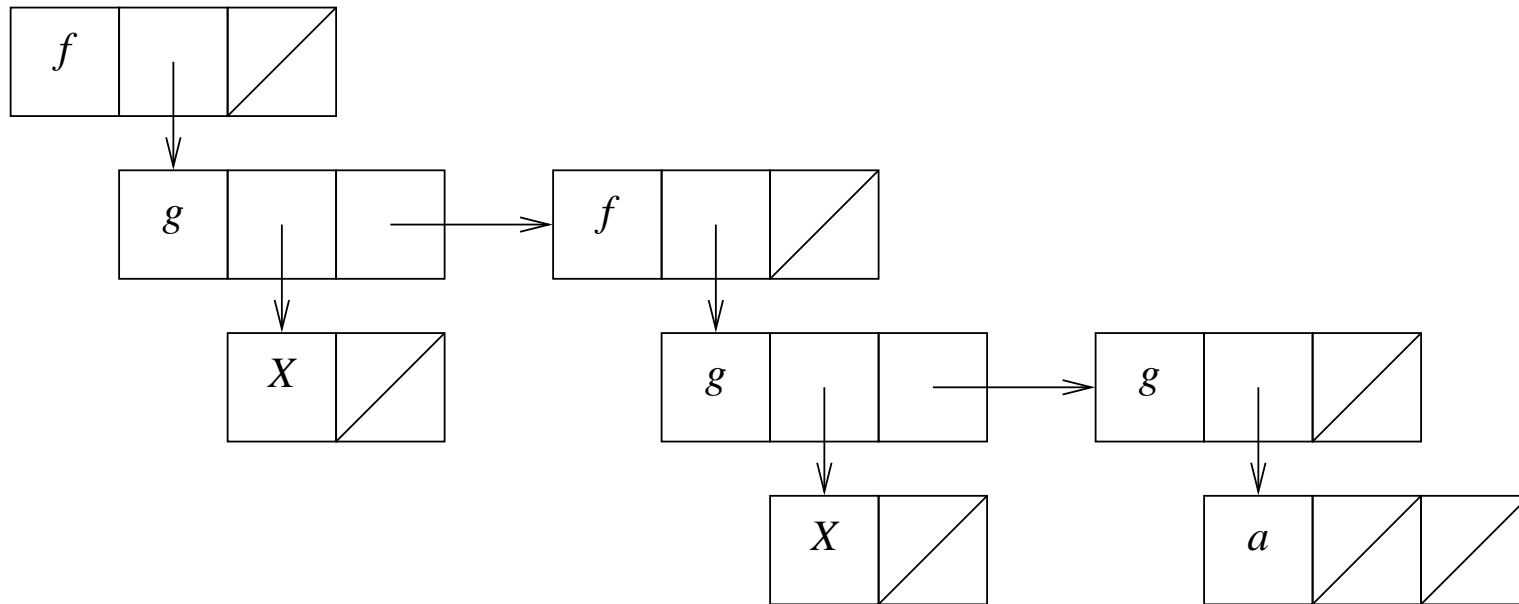
- ▶ Naive: Represent terms as strings "f(g(X), f(g(X), a))"
- ▶ More compact: "fgXfgXa"
  - Seems to be very memory-efficient!
  - But: Inconvenient for manipulation!
- ▶ Terms as ordered trees
  - Nodes are labeled with function symbols or variables
  - Successor nodes are subterms
  - Leaf nodes correspond to variables or constants
  - **Obvious** approach, used in many systems!

# Abstract Term Trees

- ▶ Example term:  $f(g(X), f(g(X), a))$

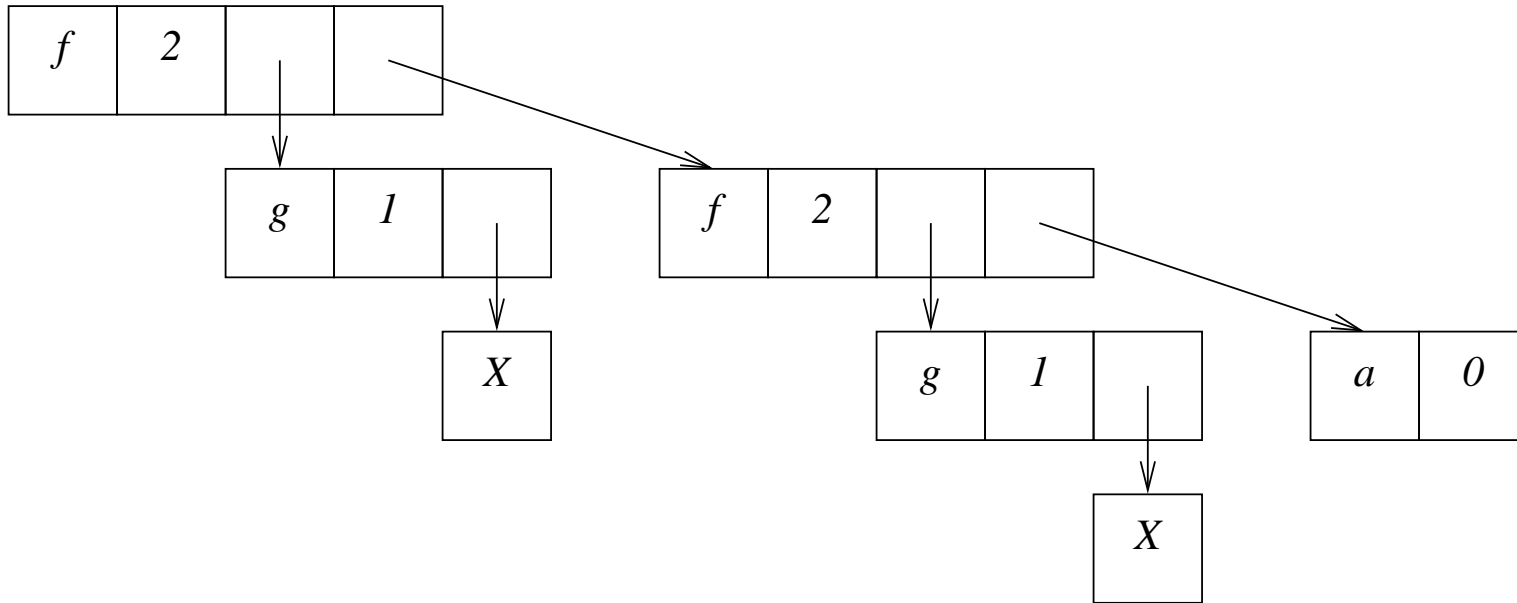


# LISP-Style Term Trees



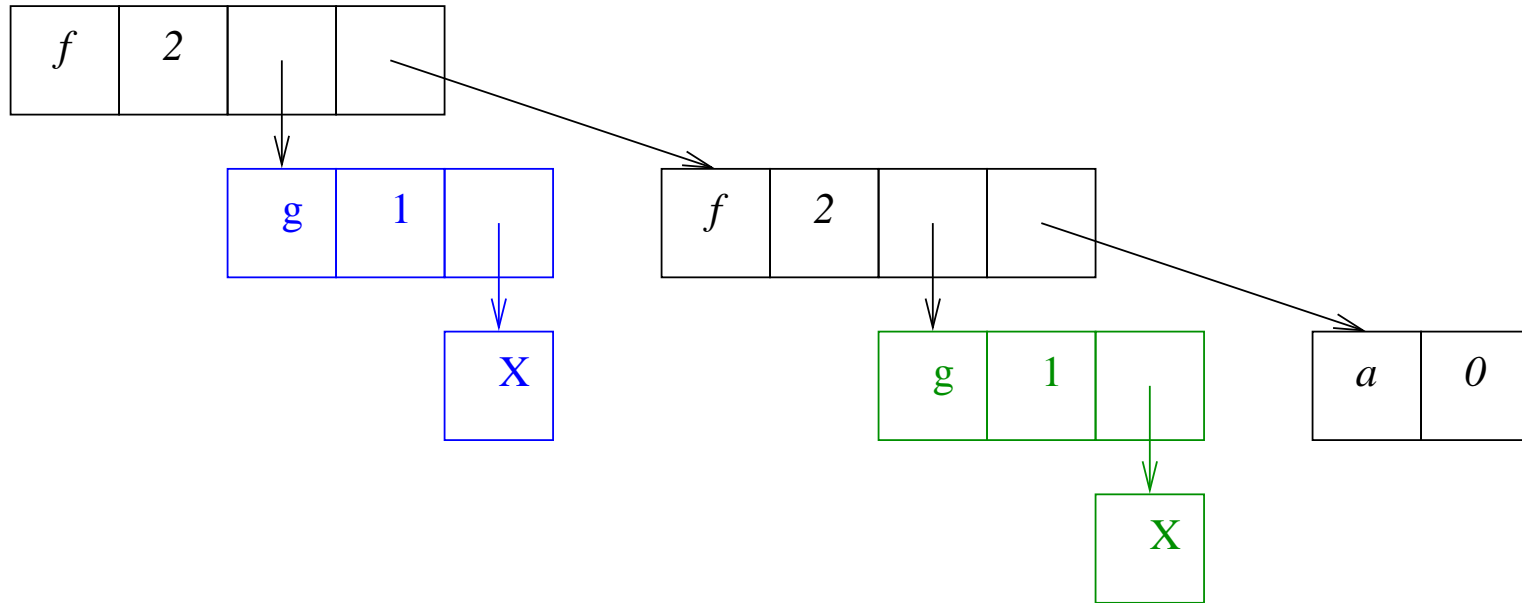
- ▶ Argument lists are represented as linked lists
- ▶ Implemented e.g. in PCL tools for DISCOUNT and Waldmeister

# C/ASM Style Term Trees



- ▶ Argument lists are represented by arrays with length
- ▶ Implemented e.g. in DISCOUNT (as an evil hack)

# C/ASM Style Term Trees

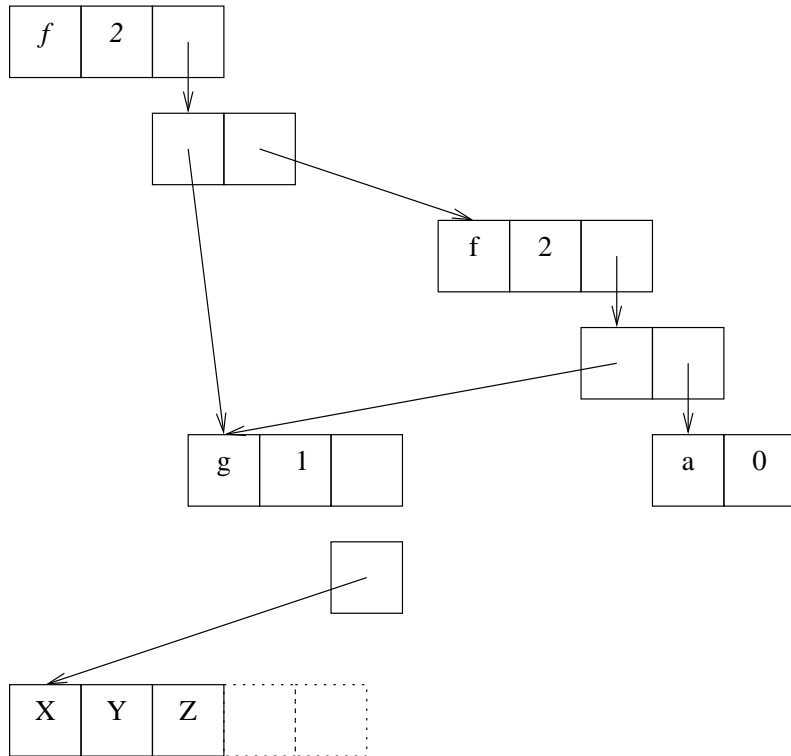


- ▶ In this version: Isomorphic subterms have isomorphic representation!

## Exercise: Term Datatype in E

- ▶ E's basic term data type is defined in `E/TERMS/cte_termtypes.h`
  - Which term representation does E use?

# Shared Terms (E)



- ▶ Idea: Consider terms not as trees, but as DAGs
  - Reuse identical parts
  - Shared variable banks (trivial)
  - Shared term banks maintained bottom-up



# Shared Terms

## ▶ Disadvantages:

- More complex
- Overhead for maintaining term bank
- Destructive changes must be avoided

## ▶ Direct Benefits:

- Saves between 80% and 99.99% of term nodes
- Consequence: We can afford to store precomputed values
  - \* Term weight
  - \* Rewrite status (see below)
  - \* Groundness flag
  - \* . . .
- Term identity: One pointer comparison!

# Literal Datatype

- ▶ See E/CLAUSES/cc1\_eqn.h
- ▶ Equations are basically pairs of terms with some properties

```
/* Basic data structure for rules, equations, literals. Terms are  
always assumed to be shared and need to be manipulated while taking  
care about references! */
```

```
typedef struct eqncell  
{  
    EqnProperties    properties; /* Positive, maximal, equational */  
    Term_p          lterm;  
    Term_p          rterm;  
    int             pos;  
    TB_p            bank;        /* Terms are from this bank */  
    struct eqncell *next;        /* For lists of equations */  
}EqnCell, *Eqn_p, **EqnRef;
```

# Clause Datatype

- ▶ See E/CLAUSES/ccl\_clause.h
- ▶ Clauses are containers with Meta-information and literal lists

```
typedef struct clause_cell
{
    long          ident;          /* Hopefully unique ident for
all clauses created during
proof run */
    SysDate       date;          /* ...at which this clause
became a demodulator */
    Eqn_p         literals;      /* List of literals */
    short         neg_lit_no;    /* Negative literals */
    short         pos_lit_no;    /* Positive literals */
    long          weight;        /* ClauseStandardWeight()
precomputed at some points in
the program */
    Eval_p        evaluations;   /* List of evaluations */
}
```

```

    ClauseProperties      properties; /* Anything we want to note at
the clause? */
...
    struct clausesetcell* set;      /* Is the clause in a set? */
    struct clause_cell*   pred;     /* For clause sets = doubly */
    struct clause_cell*   succ;     /* linked lists */
}ClauseCell, *Clause_p;

```

# Summary Day 1

- ▶ First-order logic with equality
- ▶ Superposition calculus
  - Generating inferences ("Superposition rule")
  - Rewriting
  - Subsumption
- ▶ Proof procedure
  - Basic given-clause algorithm
  - DISCOUNT Loop
- ▶ Software architecture
  - Low-level components
  - Logical datatypes

# Literature Online

- ▶ My papers are at <http://www4.informatik.tu-muenchen.de/~schulz/bibliography.html>
- ▶ The Workshop versions of Bernd Löchners LPO/KBO papers [Löc06, LÖ6] are published in the "Empirically Successful" series of Workshops. Proceedings are at [http://www.eprover.org/EVENTS/es\\_series.html](http://www.eprover.org/EVENTS/es_series.html)
  - "Things to know when implementing LPO": Proceedings of **Empirically Successful First Order Reasoning** (2004)
  - "Things to know when implementing KPO": Proceedings of **Empirically Successful Classical Automated Reasoning** (2005)
- ▶ Technical Report version of [BG94]:
  - <http://domino.mpi-inf.mpg.de/internet/reports.nsf/c125634c000710d4c12560410043ec01/c2de67aa270295ddc12560400038fcc3!OpenDocument>
  - ... or Google "Bachmair Ganzinger 91-208"

## "LUSK6" Example

```
# Problem:      In a ring, if  $x*x*x = x$  for all  $x$ 
#               in the ring, then
#                $x*y = y*x$  for all  $x,y$  in the ring.
#
# Functions:    f      : Multiplikation *
#               J      : Addition +
#               g      : Inverses
#               e      : Neutrales Element
#               a,b    : Konstanten

j (0,X)        = X.           # 0 ist a left identity for sum
j (X,0)        = X.           # 0 ist a right identity for sum
j (g (X),X)    = 0.           # there exists a left inverse for sum
j (X,g (X))    = 0.           # there exists a right inverse for sum
j (j (X,Y),Z)  = j (X,j (Y,Z)). # associativity of addition
j (X,Y)        = j(Y,X).     # commutativity of addition
f (f (X,Y),Z)  = f (X,f (Y,Z)). # associativity of multiplication
f (X,j (Y,Z))  = j (f (X,Y),f (X,Z)). # distributivity axioms
f (j (X,Y),Z)  = j (f (X,Z),f (Y,Z)). #
f (f(X,X),X)   = X.           # special hypothese:  $x*x*x = x$ 

f (a,b) != f (b,a).          # (Skolemized) theorem
```

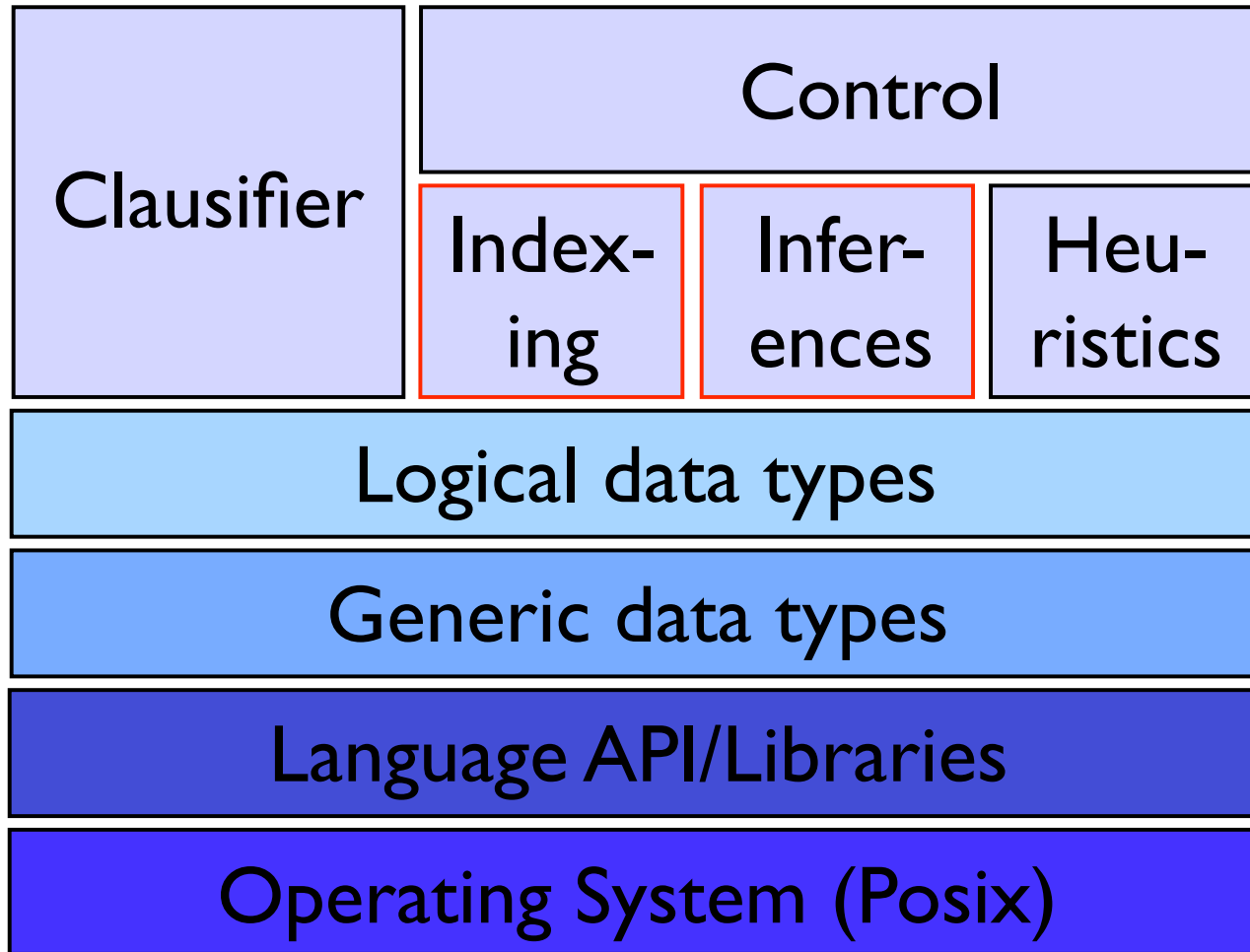
## LUSK6 in TPTP-3 syntax

```
cnf(j_neutral_left,      axiom, j(0,X)      = X).
cnf(j_neutral_right,    axiom, j(X,0)      = X).
cnf(j_inverse_left,     axiom, j(g(X),X)    = 0).
cnf(j_inverse_right,    axiom, j(X,g(X))    = 0).
cnf(j_commutates,       axiom, j(X,Y)      = j(Y,X)).
cnf(j_associates,       axiom, j(j(X,Y),Z) = j(X,j(Y,Z))).
cnf(f_associates,       axiom, f(f(X,Y),Z) = f(X,f(Y,Z))).
cnf(f_distributes_left, axiom, f(X,j(Y,Z)) = j(f(X,Y),f(X,Z))).
cnf(f_distributes_right, axiom, f(j(X,Y),Z) = j(f(X,Z),f(Y,Z))).
cnf(x_cubedequals_x,    axiom, f(f(X,X),X) = X).
```

```
fof(mult_commutates, conjecture, ![X,Y] : (f(X,Y) = f(Y,X))).
```



# Layered Architecture



# Efficient Rewriting

## ▶ Problem:

- Given term  $t$ , equations  $E = \{l_1 \simeq r_1 \dots l_n \simeq r_n\}$
- Find normal form of  $t$  w.r.t.  $E$

## ▶ Bottlenecks:

- Find applicable equations
- Check ordering constraint ( $\sigma(l) > \sigma(r)$ )

## ▶ Solutions in E:

- Cached rewriting (normal form date, pointer)
- Perfect discrimination tree indexing with age/size constraints

# Shared Terms and Cached Rewriting

- ▶ Shared terms can be long-term persistent!
- ▶ Shared terms can afford to store more information per term node!
- ▶ Hence: Store rewrite information
  - Pointer to resulting term
  - Age of youngest equation with respect to which term is in normal form
- ▶ Terms are at most rewritten once!
- ▶ Search for matching rewrite rule can exclude old equations!

# Indexing

- ▶ Quickly find inference partners in large search states
  - Replace linear search with index access
  - Especially valuable for simplifying inferences
- ▶ More concretely (or more abstractly?):
  - Given a set of terms or clauses  $S$
  - and a query term or query clause
  - and a retrieval relation  $R$
  - Build a data structure to efficiently find (all) terms or clauses  $t$  from  $S$  such that  $R(t, S)$  (the retrieval relation holds)

# Introductory Example: Text Indexing

▶ **Problem:** Given a set  $D$  of text documents, find all documents that contain a certain word  $w$

▶ Obviously correct implementation:

```
result = {}  
for doc in D  
    for word in doc  
        if w == word  
            result = result ∪ { doc }  
            break;  
return result
```

▶ Now think of **Google**. . .

- Obvious approach (linear scan through documents ) breaks down for large  $D$
- Instead: Precompiled **Index**  $I : words \rightarrow documents$
- Requirement:  $I$  **efficiently** computable for large number of words!

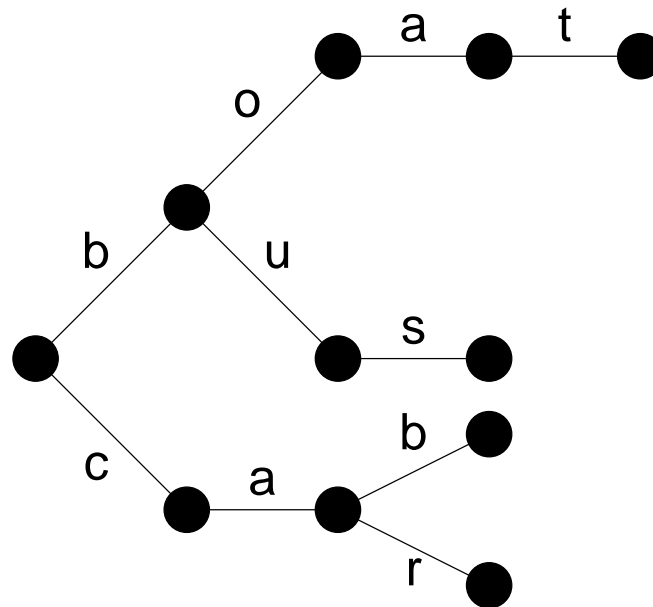
# The Trie Data Structure

- ▶ **Definition:** Let  $\Sigma$  be a finite alphabet and  $\Sigma^*$  the set of all words over  $\Sigma$ 
  - We write  $|w|$  for the length of  $w$
  - If  $u, v \in \Sigma^*$ ,  $w = uv$  is the word with **prefix**  $u$
- ▶ A **trie** is a finite **tree** whose **edges** are labelled with letters from  $\Sigma$ 
  - A **node** represents a set of words with a common prefix (defined by the labels on the path from the root to the node)
  - A **leaf** represents a single word
  - The whole **trie** represents the set of words at its leaves
  - Dually, for each set of words  $S$  (such that no word is the prefix of another), there is a unique trie  $T$
- ▶ **Fact:** Finding the leaf representing  $w$  in  $T$  (if any) can be done in  $O(|w|)$ 
  - **This is independent of the size of  $S$ !**
  - Inserting and deleting of elements is just as fast

# Trie Example

- ▶ Consider  $\Sigma = \{a, b, \dots, z\}$  and  $S = \{car, cab, bus, boat\}$

- ▶ The trie for  $S$  is:



- ▶ Tries can be built incrementally
- ▶ We can store extra information at nodes/leaves
  - E.g. all documents in which *boat* occurs
  - Retrieving this information is fast and simple

# Indexing Techniques for Theorem Provers

- ▶ **Term Indexing** standard technique for high performance theorem provers
  - Preprocess term sets into **index**
  - Return terms in a certain relation to a **query term**
    - \* Matches query term (find generalizations)
    - \* Matched by query term (find specializations)
- ▶ Perfect indexing:
  - Returns exactly the desired set of terms
  - May even return substitution
- ▶ Non-perfect indexing:
  - Returns **candidates** (superset of desired terms)
  - Separate test if candidate is solution



# Frequent Operations

- ▶ Let  $S$  be a set of clauses
- ▶ Given term  $t$ , find an applicable rewrite rule in  $S$ 
  - Forward rewriting
  - Reduced to: Given  $t$ , find  $l \simeq r \in S$  such that  $l\sigma = t$  for some  $\sigma$
  - Find generalizations
- ▶ Given  $l \rightarrow r$ , find all rewritable clauses in  $S$ 
  - Backward rewriting
  - Reduced to: Given  $l$ , find  $t$  such that  $C|_p\sigma = l$
  - Find instances
- ▶ Given  $C$ , find a subsuming clause in  $S$ 
  - Forward subsumption
  - Not easily reduced. . .
  - Backward subsumption analogous

# Classification of Indexing Techniques

## ▶ Perfect indexing

- The index returns **exactly** the elements that fulfill the retrieval condition
- Examples:
  - \* Perfect discrimination trees
  - \* Substitution trees
  - \* Context trees

## ▶ Non-perfect indexing:

- The index returns a **superset** of the elements that fulfill the retrieval condition
- Retrieval condition has to be verified
- Examples:
  - \* (Non-perfect) discrimination trees
  - \* (Non-perfect) Path indexing
  - \* Top-symbol hashing
  - \* Feature vector-indexing

# The Given Clause Algorithm

$U$ : Unprocessed (passive) clauses (initially Specification)

$P$ : Processed (active) clauses (initially: empty )

while  $U \neq \{\}$

$g = \text{delete\_best}(U)$

$g = \text{simplify}(g, P)$

    if  $g == \square$

        SUCCESS, Proof found

    if  $g$  is not redundant w.r.t.  $P$

$T = \{c \in P \mid c \text{ redundant or simplifiable w.r.t. } g\}$

$P = (P \setminus T) \cup \{g\}$

$T = T \cup \text{generate}(g, P)$

        foreach  $c \in T$

$c = \text{cheap\_simplify}(c, P)$

            if  $c$  is not trivial

$U = U \cup \{c\}$

SUCCESS, original  $U$  is satisfiable

Typically,  $|U| \sim |P|^2$  and  $|U| \approx \sum |T|$

# The Given Clause Algorithm

$U$ : Unprocessed (passive) clauses (initially Specification)

$P$ : Processed (active) clauses (initially: empty )

while  $U \neq \{\}$

$g = \text{delete\_best}(U)$

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        foreach  $c \in T$

$c = \text{cheap\_simplify}(c, P)$

            if  $c$  is not trivial

$U = U \cup \{c\}$

SUCCESS, original  $U$  is satisfiable

Simplification of new clauses is bottleneck

# Sequential Search for Forward Rewriting

- ▶ Given  $t$ , find  $l \simeq r \in S$  such that  $l\sigma = t$  for some  $\sigma$
- ▶ Naive implementation (e.g. DISCOUNT):

```
function find_matching_rule( $t, S$ )  
  for  $l \simeq r \in S$   
     $\sigma = \text{match}(l, t)$   
    if  $\sigma$  and  $l\sigma > r\sigma$   
      return  $(\sigma, l \simeq r)$ 
```

- ▶ Remark: We assume that for unorientable  $l \simeq r$ , both  $l \simeq r$  and  $r \simeq l$  are in  $S$

# Conventional Matching

match( $s, t$ )

return match\_list( $[s], [t], \{\}$ )

match\_list( $ls, lt, \sigma$ )

while  $ls \neq []$

$s = \text{head}(ls)$

$t = \text{head}(lt)$

    if  $s == X \in V$

        if  $X \leftarrow t' \in \sigma$

            if  $t \neq t'$  return FAIL

        else

$\sigma = \sigma \cup \{X \leftarrow t\}$

    else if  $t == X \in V$  return FAIL

    else

        let  $s = f(s_1, \dots, s_n)$

        let  $t = g(t_1, \dots, t_m)$

        if  $f \neq g$  return FAIL /\* Otherwise  $n = m!$  \*/

$ls = \text{append}(\text{tail}(ls), [s_1, \dots, s_n])$

$lt = \text{append}(\text{tail}(lt), [t_1, \dots, t_m])$

return  $\sigma$

# The Size of the Problem

- ▶ Example LUSK6:
  - Run time with E on 1GHz Powerbook: 1.7 seconds
  - Final size of  $P$ : 265 clauses (processed: 1542)
  - Final size of  $U$ : 26154 clauses
  - Approximately 150,000 **successful** rewrite steps
  - Naive implementation:  $\approx$  50-150 times more match attempts!
  - $\approx$  100 machine instructions/match attempt
  
- ▶ Hard examples:
  - Several hours on 3+GHz machines
  - Billions of rewrite attempts
  
- ▶ Naive implementations don't cut it!

# Top Symbol Hashing

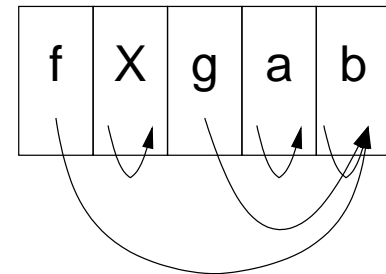
- ▶ Simple, non-perfect indexing method for (forward-) rewriting
- ▶ **Idea:** If  $t = f(t_1, \dots, t_n)$  ( $n \geq 0$ ), then any  $s$  that matches  $t$  has to start with  $f$ 
  - $top(t) = f$  is called the **top symbol** of  $t$
- ▶ Implementation:
  - Organize  $S = \cup S_f$  with  $S_f = \{l \simeq r \in S \mid top(l) = f\}$
  - For non-variable query term  $t$ , test only rewrite rules from  $S_{top(t)}$
- ▶ Efficiency depends on problem composition
  - Few function symbols: Little improvement
  - Large signatures: Huge gain
  - Typically: Speed-up factor 5-15 for matching



# String Terms and Flat Terms

- ▶ Terms are (conceptually) **ordered trees**
  - Recursive data structure
  - But: Conventional matching always does left-right traversal
  - Many other operations do likewise
- ▶ Alternative representation: **String terms**
  - $f(X, g(a, b))$  already is a string. . .
  - If arity of function symbols is fixed, we can drop braces:  $fXgab$
  - Left-right iteration is much faster (and simpler) for string terms

- ▶ **Flat terms**: Like string terms, but with **term end pointers**



- Allows fast jumping over subterms for matching

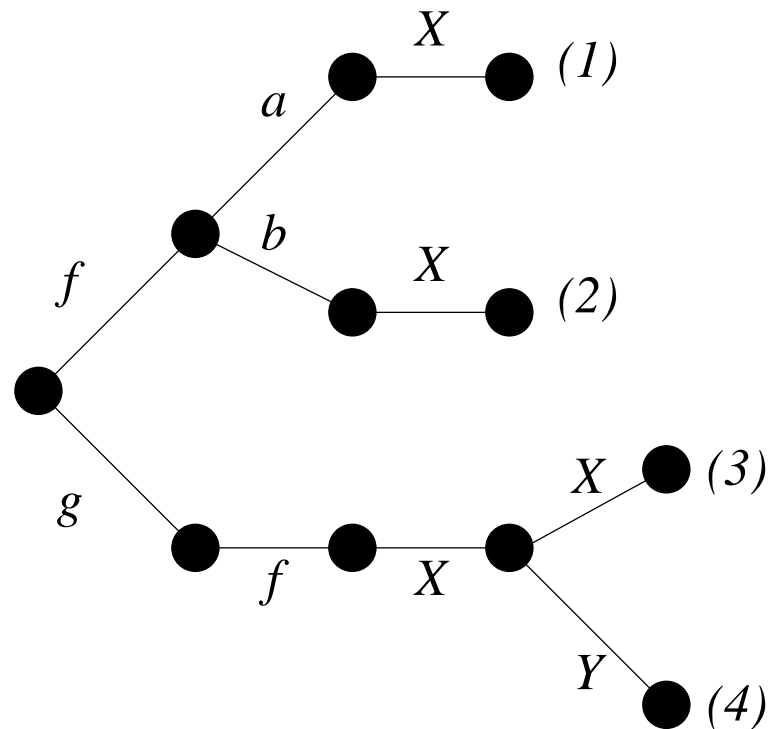
# Perfect discrimination tree indexing

- ▶ Generalization of top symbol hashing
- ▶ Idea: Share **common prefixes** of terms in string representation
  - Represent terms as **strings**
  - Store string terms (left hand sides of rules) in **trie** (perfect discrimination tree)
  - Recursively traverse trie to find matching terms for a query:
    - \* At each node, follow all compatible vertices in turn
    - \* If following a variable branch, add binding for variable
    - \* If no valid possibility, backtrack to last open choice point
    - \* If leaf is reached, report match
- ▶ Currently most frequently used indexing technique
  - E (rewriting, unit subsumption)
  - Vampire (rewriting, unit- and non-unit subsumption (as code trees))
  - Waldmeister (rewriting, unit subsumption, paramodulation)
  - Gandalf (rewriting, subsumption)
  - . . . .

## Example

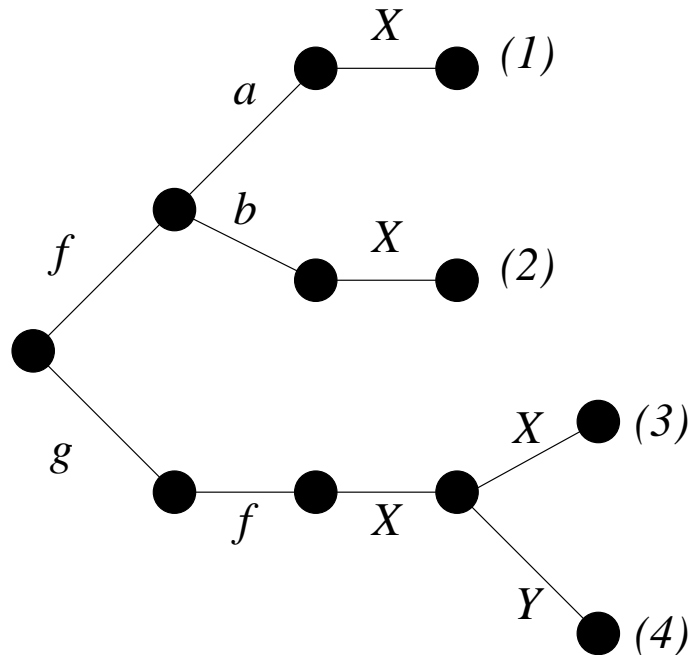
- ▶ Consider  $S = \{(1)f(a, X) \simeq a, (2)f(b, X) \simeq X,$   
 $(3)g(f(X, X)) \simeq f(Y, X), (4)g(f(X, Y)) \simeq g(X)\}$
- String representation of left hand sides:  $faX, fbX, gfXX, gfXY$

– Corresponding trie:



Find matching rule for  $g(f(a, g(b)))$

## Example Continued



- ▶ Start with  $g(f(a, g(b)))$ , root node,  $\sigma = \{\}$

$g(f(a, g(b)))$  Follow  $g$  vertex

$g(f(a, g(b)))$  Follow  $f$  vertex

$g(f(a, g(b)))$  Follow  $X$  vertex,  $\sigma = \{X \leftarrow a\}$ , jump over  $a$

$g(f(a, g(b)))$

- Follow  $X$  vertex - **Conflict!**  $X$  already bound to  $a$
- Follow  $Y$ ,  $\sigma = \{X \leftarrow a, Y \leftarrow g(b)\}$ , jump over  $g(b)$  **Rule 4 matches**

# Subsumption Indexing

- ▶ Subsumption: Important simplification technique for first-order reasoning
  - Drop less general (redundant) clauses
  - Keep more general clause
- ▶ Problem: Efficiently detecting subsumed clauses
  - Individual clause-clause subsumption is in NP
  - Large number of subsumption relations must be tested
- ▶ Major Approach: Indexing
  - Use precompiled data structures to efficiently select
    - \* subsuming clauses (**forward subsumption**)
    - \* subsumes clause (**backward subsumption**)
  - from large (and fairly static) clause sets
- ▶ Usual: Different and complex indexing approaches for forward- and backward subsumption

# Subsumption

- ▶ Idea: Only keep the most general clauses
  - If one clause is **subsumed** by another, discard it
- ▶ Formally: A clause  $C$  subsumes  $C'$  if:
  - There exists a substitution  $\sigma$  such that  $C\sigma \subseteq C'$
  - Note: In that case  $C \models C'$
  - $\subseteq$  **usually** is the **multi-subset relation**
- ▶ Examples:
  - $p(X)$  subsumes  $p(a) \vee q(f(X), a)$  ( $\sigma = \{X \leftarrow a\}$ )
  - $p(X) \vee p(Y)$  does not multi-set-subsume  $p(a) \vee q(f(X), a)$
  - $q(X, Y) \vee q(X, a)$  subsumes  $q(a, a) \vee q(a, b)$
- ▶ Subsumption is hard (NP-complete)
  - $n!$  permutations in non-equational clause with  $n$  literals
  - $n!2^n$  permutations in equational clause with  $n$  literals

# Forward- and Backward Subsumption

- ▶ Assume a **set** of clauses  $P$  and a given clause  $p$
- ▶ Forward subsumption: Is there **any** clause in  $P$  that subsumes  $g$ ?
- ▶ Backward subsumption: Find/remove **all** clauses in  $P$  subsumed by  $g$
- ▶ Notice that these are **clause–clause set** operations
- ▶ **Naive implementation:** Sequence of clause-clause operations
  - Good implementation can speed up (average case) individual subsumption
  - Number of attempts still very high
- ▶ Smarter: Avoid many of the subsumption calls up front
  - Use **indexing techniques** to reduce number of candidates

# Feature Vector Indexing

- ▶ New **clause indexing** technique
  - Not lifted from term indexing
- ▶ Advantages:
  - Small index (memory footprint)
  - Same index for forward- and backward subsumption
  - Very simple
  - Efficient in practice
  - Variants for different subsumption relations
- ▶ Disadvantages:
  - Non-perfect
  - Requires fixed signature for optimal performance

**How does it work?**



# Properties of the Subsumption Relation

## Definitions:

- Let  $C$  and  $C'$  be clauses
- $C^+$  is the (multi-)set (a clause) of positive literals in  $C$
- $C^-$  is the (multi-)set of negative literals in  $C$
- $|C|_f$  is the number of occurrences of (function or predicate) symbol  $f$  in  $C$

## Facts: If $C$ subsumes $C'$ , then

- $|C^+| \leq |C'^+|$
- $|C^-| \leq |C'^-|$
- $|C^+|_f \leq |C'^+|_f$  for all  $f$
- $|C^-|_f \leq |C'^-|_f$  for all  $f$
- (Similar results exist for term depths)
- The same holds for all linear combination of these features

## Remark: Composite criteria are often used to detect subsumption failure early

- $|C| \leq |C'|$  ( $C$  cannot have more literals than  $C'$ )
- $\sum_{f \in F} |C|_f \leq \sum_{f \in F} |C'|_f$  ( $C$  cannot have more symbols than  $C'$ )

# Feature Vectors

## Definitions:

- A **feature function**  $f$  is a function from the set of clauses to  $\mathbf{N}$
- $f$  is **subsumption-compatible**, if  $C$  subsumes  $C'$  implies  $f(C) \leq f(C')$
- A (subsumption-compatible) **feature vector function**  $F$  is a function from the set of clauses to  $\mathbf{N}^n$  such that  $\Pi_n^i \circ F$  (the projection of  $F$  to the  $i$ th component) is a subsumption-compatible feature function
- If  $v_1$  and  $v_2$  are feature vectors, we write  $v_1 \leq_s v_2$ , if  $v_1[i] \leq v_2[i]$  for all  $i$ .

## Fact:

- Assume  $F$  is a (subsumption-compatible) feature vector function
- Assume  $C$  subsumes  $C'$
- By construction,  $F(C) \leq_s F(C')$

## Basic Principle of Feature Vector Indexing:

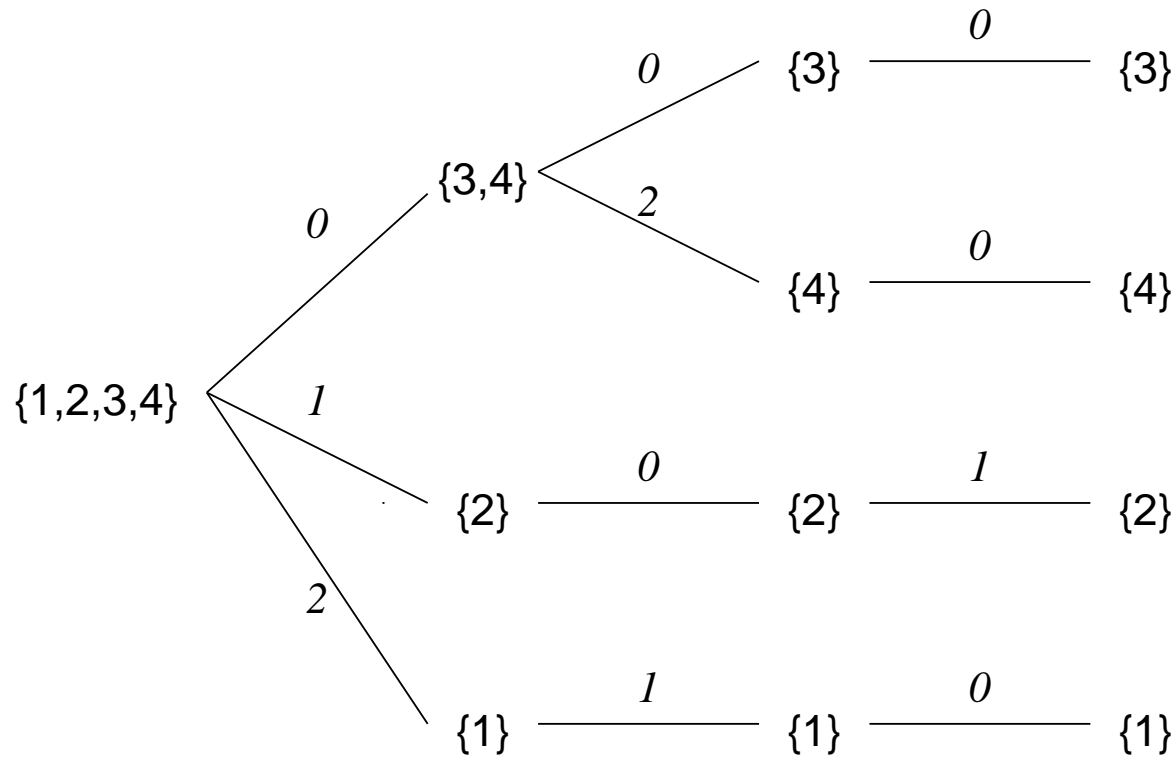
- For forward-subsumption:  $candFS_F(P, g) = \{c \in P \mid F(c) \leq_s F(g)\}$
- For backward-subsumption:  $candBS_F(P, g) = \{c \in P \mid F(g) \leq_s F(c)\}$

# Feature Vector Indexing

- ▶ **Aim:** Efficiently compute  $candFS_F(P, g)$  and  $candBS_F(P, g)$
- ▶ **Solution:** Frequency vectors for  $P$  are compiled into a **trie**, clauses are stored in leaves
  - Tree of depth  $n$  (number of features in vector)
  - Nodes at depth  $d$  split according to feature  $F(C)[d]$  (one successor per value)
  - All vectors with value  $F(C)[d] = k$  associated with corresponding subtree
  - Construction continues recursively
- ▶ **Example:** Assume  $F(C) := \langle |C^+|_a, |C^+|_f, |C^-|_b \rangle$ 
  - **Clause set**  $P = \{1, 2, 3, 4\}$  with
    1.  $F(p(a) \vee p(f(a))) = \langle 2, 1, 0 \rangle$
    2.  $F(p(a) \vee \neg p(b)) = \langle 1, 0, 1 \rangle$
    3.  $F(\neg p(a) \vee p(b)) = \langle 0, 0, 0 \rangle$
    4.  $F(p(X) \vee p(f(f(b)))) = \langle 0, 2, 0 \rangle$
  - **Query**  $g = p(f(a))$ 
    - \*  $F(g) = \langle 1, 1, 0 \rangle$

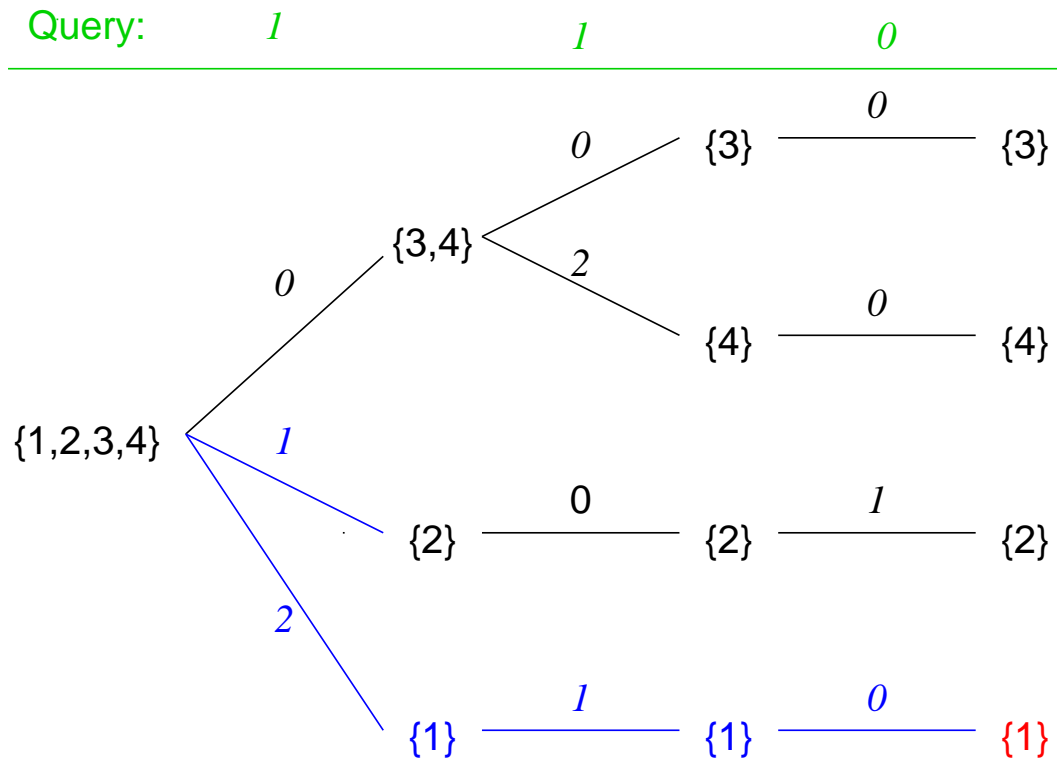
## Example Index

1.  $F(p(a) \vee p(f(a))) = \langle 2, 1, 0 \rangle$
2.  $F(p(a) \vee \neg p(b)) = \langle 1, 0, 1 \rangle$
3.  $F(\neg p(a) \vee p(b)) = \langle 0, 0, 0 \rangle$
4.  $F(p(X) \vee p(f(f(b)))) = \langle 0, 2, 0 \rangle$



## Example: Backward Subsumption

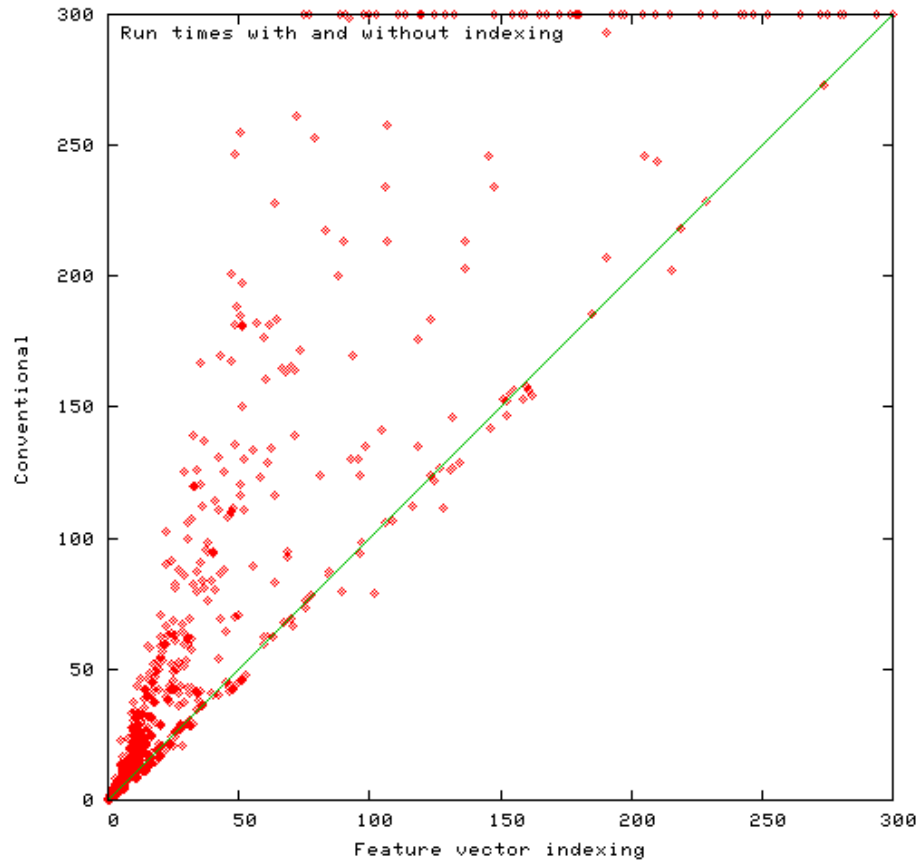
- ▶ Algorithm: At each node, only follow branches with larger or equal feature values



- ▶ Result: Just one subsumption candidate for  $p(f(a))$

# Performance 1

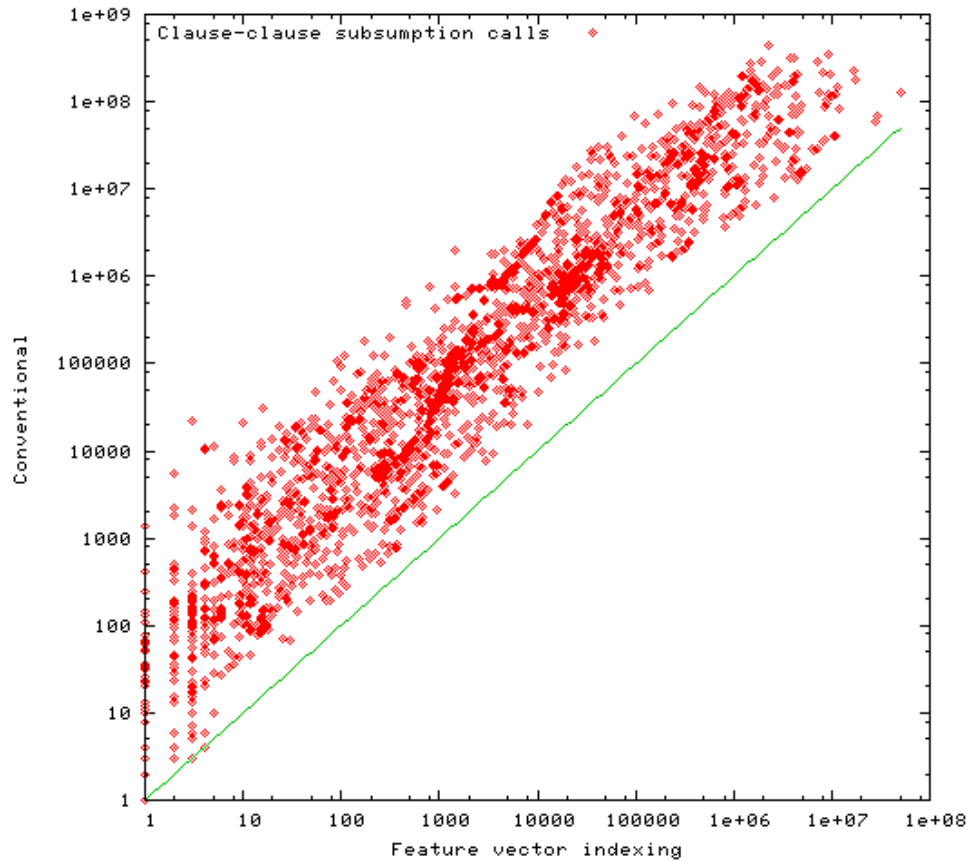
- ▶ Tested on 5180 examples from TPTP 2.5.1
  - Subsumption-heavy search strategy (contextual literal cutting)
  - Max. 75 features, 300MHz SUN Ultra 60, 300s time limit



- ▶ Speedup ca. 40%, overhead usually insignificant, 2717 vs. 2671 solutions found

## Performance 2

- ▶ Number of subsumption attempts (notice double log scale)



- ▶ Average reduction: 1 : 60, max: 1 : 8000(1 :  $\infty$ )

# Literature on Indexing

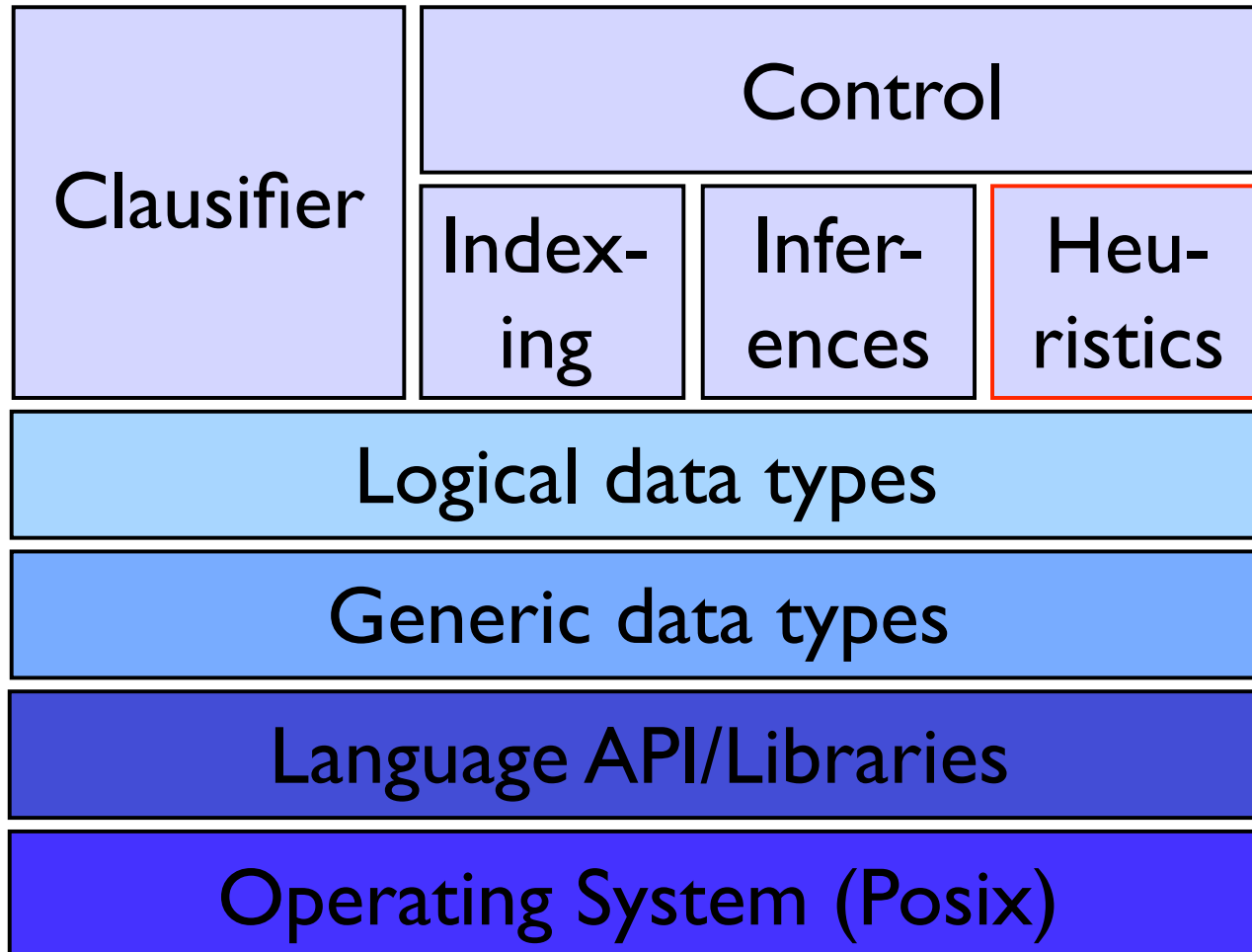
- ▶ Overview: [Gra95, SRV01]
- ▶ Classic paper: [McC92]
- ▶ Comparisons (for rewriting): [NHRV01]
- ▶ Feature vector indexing: [Sch04a]



## Exercise: Unification

- ▶ E's unification code is `SubstComputeMgu()` in `E/TERMS/cte_match_mgu_1-1.[hc]`
  - Read and understand the code
  - Unification is broken down into subtasks
  - Subtasks are stored in a particular order
  - Why? Experiment with different orders!

# Layered Architecture



# Don't-care-Nondeterminism $\equiv$ Chances for Heuristics

- ▶ Important choice points for E:
  - Simplification ordering
  - Clause selection
  - Literal selection
  
- ▶ Other choice points:
  - Choice of rewrite relation (usually strongest, don't care **which** normal form)
  - Application of rewrite relation to terms (leftmost-innermost, strongly suggested by shared terms)

# Simplification Orderings

- ▶ Implemented: Knuth-Bendix-Orderings, Lexicographic Path Orderings
- ▶ Precedence: Fully user defined or simple algorithms
  - **Sorted by arity** (higher arity  $\rightarrow$  larger)
  - **Sorted by arity, but unary first**
  - Sorted by inverse arity
  - Sorted by frequency of appearance in axioms
  - . . .
- ▶ Weights for KBO: Similar simple algorithms (constant weights (optionally weight 0 for maximal symbol), arity, position in precedence . . . )
- ▶ No good automatic selection of orderings yet – auto mode switches between two simple KBO schemes

# Clause selection

- ▶ Most important choice point (?)
- ▶ Probably also hardest choice (find best clause among millions)
- ▶ Implementation in E: Multiple priority queues sorted by heuristic evaluation and strategy-defined priority
- ▶ Selection in weighted round-robin-scheme (generalizes **pick-given** ratio)
- ▶ Example:  $8 * \text{Refinedweight}(\text{PreferGoals}, 1, 2, 2, 3, 0.8),$   
 $8 * \text{Refinedweight}(\text{PreferNonGoals}, 2, 1, 2, 3, 0.8),$   
 $1 * \text{Clauseweight}(\text{ConstPrio}, 1, 1, 0.7),$   
 $1 * \text{FIFOWeight}(\text{ByNegLitDist})$
- ▶ Big win: Goal directed search
  - Symbols in the goal have low (=good) weights
  - Other symbols have increasingly large weight based on linking distance

# Literal Selection

- ▶ Problem: Which literals should be selected for inferences in a clause?
- ▶ Ideas:
  - Select hard literals first (if we cannot solve this, the clause is useless)
  - Select small literals (fewer possible overlaps)
  - Select ground literals (no instantiation, most unit-overlaps eliminated by rewriting)
  - Propagate inference literals to children clauses (inheritance)
- ▶ Problem: Should we always select literals if possible?
  - Only select if no unique maximal literal exists
  - Do not select in conditional rewrite rules
- ▶ Surprisingly successful: Additional selection of maximal positive literals
- ▶ See E source code for large number of things we have tried. . .

# Literature on other Systems

- ▶ Real (saturating) provers: [LH02, RV02, Sch02, Wei01, WSH<sup>+</sup>07, Sti92, Sti89, LS01b]
  
- ▶ Significant alternative approaches:
  - DCTP [SL01, LS01a, LS02],
  - Model elimination: SETHEO [LSBB92, MIL<sup>+</sup>97], leanCOP [OB03, Ott08]
  - Instantiation-Based Reasoning: iProver: [Kor08, Kor09]
  - Model Evolution: Darwin [BFT06]

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