A Survey of Program Termination: Practical and Theoretical Challenges

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Instructive Example

- Consider the following order-5 recurrence:

$$u_{n+5} = -\frac{19}{25}u_{n+4} - \frac{114}{125}u_{n+3} + \frac{114}{125}u_{n+2} + \frac{19}{25}u_{n+1} + u_n$$

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- This is simple, with characteristic roots $1, \lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}$, where

$$\lambda_1 = \frac{-3+4i}{5} \text{ and } \lambda_2 = \frac{-7+24i}{25}$$

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- For suitably chosen initial values we have

$$u_n = \tfrac{33}{8} + \lambda_1^n + \overline{\lambda_1^n} + 2\lambda_2^n + 2\overline{\lambda_2^n}$$

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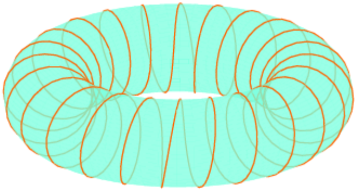

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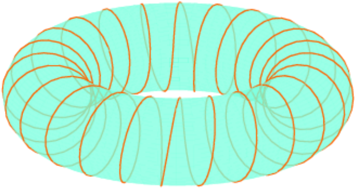

- Point $(-1, -1)$ does not lie on helix.

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- But what about $u_n - \frac{1}{2^n}$?