Refinement based Algorithm Development

with Isabelle/HOL

Peter Lammich

Fakultät für Informatik
Technische Universität München

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Introduction

- Why Program Verification

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  - See previous lectures ;)

Short Poll

Raise your hand if you know/ have heard of

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- Parametricity (eg Theorems for Free!)

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- Separation Logic

Overview

- Refinement based approach to algorithm development
  - From pseudocode to implementation
- Everything verified within Isabelle/HOL
  - We do not trust any other tools
- Edmonds-Karp Maxflow algorithm as running example
  - Approach has been used for many formalizations. Highlights:
    MUNTA: Timed Automata Model Checker
    GRAT: SAT solver certification
    CAVA: LTL model checker

Lecture Material:
`http://www21.in.tum.de/~lammich/vtsa2018_isabelle.tgz`

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  - Only this kernel can prove theorems
  - Large set of tools on top of kernel
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- Archive of Formal Proofs `https://www.isa-afp.org/`
  - Large set of theories readily available
  - Maintained to run with latest Isabelle version

Isabelle/HOL

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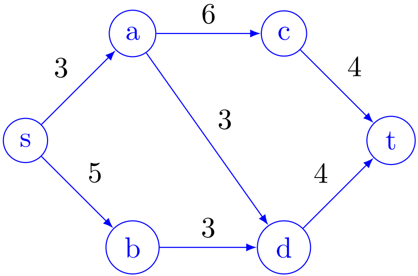
- Trying to present ideas independent of Isabelle
- But many examples and demos in Isabelle

Flow Networks and Flows

- Flow Network
  - Directed graph
  - Edges annotated with capacity
  - Distinguished source and sink node
- Flow
  - Generated only at source
  - Consumed only at sink
  - Must not exceed edge capacities

Finding Maximum Flow



- Start with empty flow

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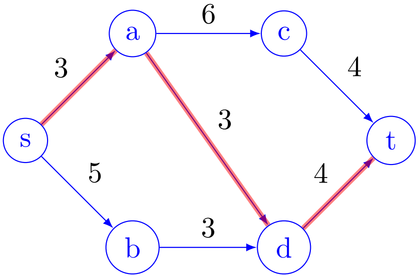

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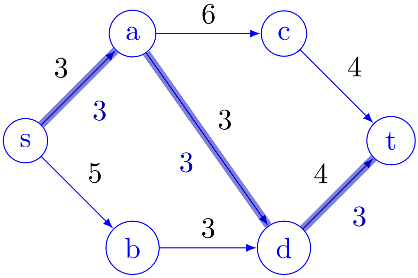

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    - Increase flow

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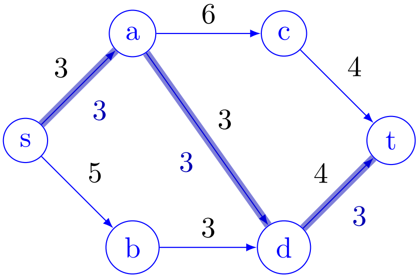

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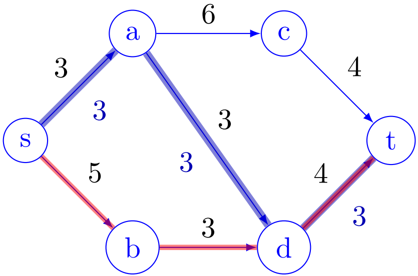

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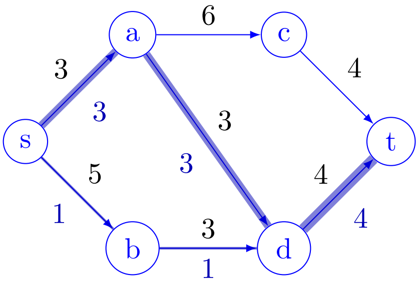

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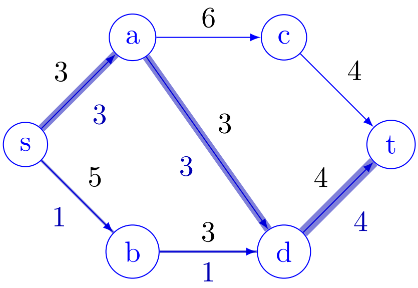

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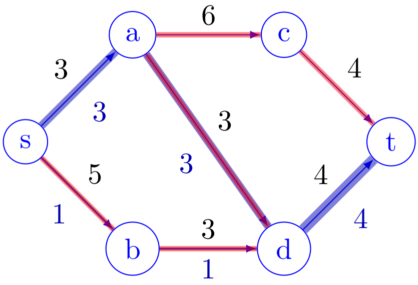

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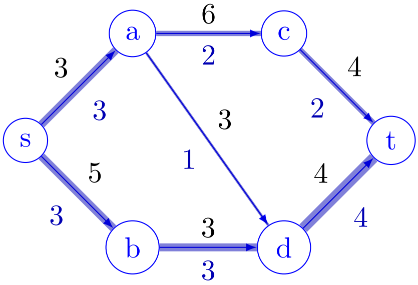

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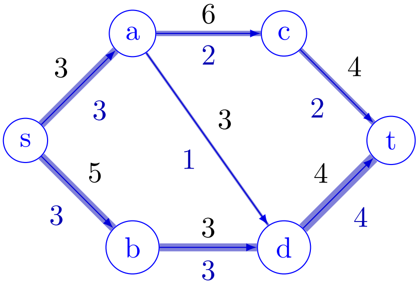

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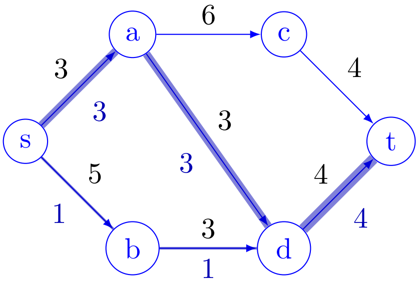
- Flow is maximal now

- Flow that can be moved between nodes
  - By increasing or taking back flow



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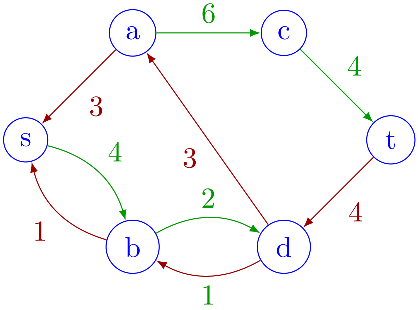

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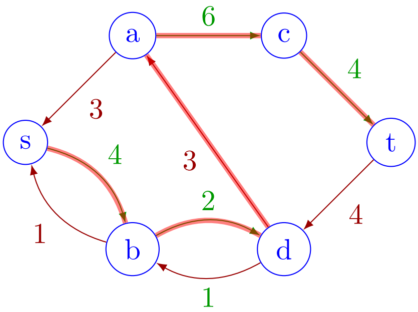
- Augmenting path: s-t path in residual graph



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- Edmonds/Karp: choose shortest augmenting path
  - $O(VE)$ iterations for real-valued capacities
  - Using BFS to find path: $O(VE^2)$ algorithm

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- Work on residual graphs instead of flows
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- Use BFS to find shortest path
- Data structures for graph, augmenting path, BFS algorithm
- Write Standard-ML program (Or Scala, Haskell, Ocaml, ...)

Data Structures

- Residual Graph
  - Operations: successors of node, capacity of edge
  - Nodes by natural numbers from $\{0..<N\}$
  - Adjacency matrix by array ($\mathit{capacity}[N][N]$)
  - Adjacency map by array $(\mathit{node\ list})[N]$

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    - Or use $\mathit{int}[N]$ and map None to -1
  - Current and next set: *node list*
- Augmenting path: *node list*
  - Could use predecessor map directly
  - but converting to list gives cleaner interface, and is no bottleneck

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- Informally, correctness argument given on abstract algorithm

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- Then, we described how to implement the parts of the algorithm
- And concluded: Abstract algorithm is correct and implemented correctly
  $\Longrightarrow$ implementation is correct

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- Show that it corresponds to abstract algorithm
- By transitivity, argue that implementation returns maxflow
  impl $\leq$ *abstract algo* $\leq$ *specification*

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- Standard modular design patterns apply
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Why should one care about BFS algorithm or non-overflow of adjacency matrix array access, when demonstrating the abstract idea of Edmonds-Karp algorithm?

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- Nondeterminism
  - Cannot determine actual shortest path in abstract algorithm
  - There may be many, and implementation details of BFS decide which one is returned
  - Abstractly: nondeterministically choose among all possibilities

Interlude: Monads for Programming

- A monad is a type $'a\ M$ with operations

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- Monad laws
  - $\textbf{do}\ \{\ x \leftarrow \textbf{return}\ v;\ f\ x\ \} = f\ v$
  - $\textbf{do}\ \{\ x \leftarrow m;\ \textbf{return}\ x\ \} = m$
  - $\textbf{do}\ \{\ y \leftarrow \textbf{do}\ \{x \leftarrow m;\ f\ x\ \};\ g\ y\ \} = \textbf{do}\ \{\ x \leftarrow m;\ y \leftarrow f\ x;\ g\ y\ \}$

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  - $\textbf{do}\ \{\ \ldots;\ \textbf{let}\ (x_1, x_2, x_3) = a;\ \ldots\ \}$
- Any HOL function, and its syntax

$$
\begin{array}{l}
\textbf{do}\ \{ \\
\quad x {\leftarrow} m; \\
\quad \textbf{if}\ x{<}0\ \textbf{then return}\ (-1\ ) \\
\quad \textbf{else if}\ x{=}0\ \textbf{then return}\ 0 \\
\quad \textbf{else if}\ x{>}0\ \textbf{then return}\ 1 \\
\}
\end{array}
$$

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  - Computations that can fail: $M = option$

    return $x = Some\ x$
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    - Example: **do** $\{\ \textbf{if}\ x{=}0\ \textbf{then}\ fail\ \textbf{else}\ \textbf{return}\ (1\ /\ x)\ \}$
  - Nondeterministic computations $M{=}set$

    $\textbf{return}\ x = \{x\}$
    $bind\ m_1\ m_2 = \bigcup \{\ m_2\ x \mid x.\ x{\in}m_1\ \}$
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Monads for Programming

- Monad models sequential execution
  - **do** $\{\ x \leftarrow m;\ f\ x\ \}$ First execute m, then f
- More functionality can be added by structure of type $'a\ M$
  - Computations that can fail: $M = option$

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    - Example: **do** $\{\ \mathbf{if}\ x{=}0\ \mathbf{then}\ fail\ \mathbf{else}\ \mathbf{return}\ (1\ /\ x)\ \}$
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  - Many more: exceptions, state, output, probability, ...

The *nres* Monad

- Combines failure and nondeterminism monad. $M = nres$ where

  $\textbf{datatype}\ 'a\ nres = FAIL \mid RES\ 'a\ set$

  $$\begin{aligned}
  &\textbf{return}\ x &&= RES\ \{x\} \\
  &bind\ FAIL\ f &&= FAIL \\
  &bind\ (RES\ X)\ f &&= Sup\ \{f\,x \mid x.\ x{\in}X\}
  \end{aligned}$$

  where

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- Derived combinators

  $$\begin{aligned}
  \textbf{spec}\ x.\ \Phi\ x &= \mathit{RES}\ \{x.\ \Phi\ x\} \\
  \textbf{assert}\ \Phi &= \textbf{if}\ \Phi\ \textbf{then return}\ ()\ \textbf{else}\ \mathit{FAIL} \\
  \textbf{select}\ p.\ \Phi\ p &= \textbf{if}\ \exists x.\ \Phi\ x\ \textbf{then}\ \mathit{RES}\ \{\mathit{Some}\ x \mid x.\ \Phi\ x\}\ \textbf{else return}\ \mathit{None}
  \end{aligned}$$

Structural Recursion

- Generalized fold combinator

  $\mathit{nfoldli}\ []\ c\ f\ s = \textbf{return}\ s$

  $\mathit{nfoldli}\ (x \# \mathit{ls})\ c\ f\ s =$

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  $\textbf{foreach}\ S\ f\ \sigma = \textbf{do}\ \{$
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  - Warning: Actual implementation of **foreach** and friends suffers from legacy problems. But works nicely at the surface!

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- Recursion via fixed-point construction

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  $\mathbf{while}_T\ b\ f \equiv \mathbf{rec}_T\ W.\ (\lambda s.\ \mathbf{if}\ b\ s\ \mathbf{then}\ f\ s \gg= W\ \mathbf{else\ return}\ s)$
- Program is ordinary term in HOL (shallow embedding)

Examples

Select item from (non-empty) set S :

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$$
\begin{array}{l}
\mathit{sum\_up}\ S = \textbf{do}\ \{ \\
\quad (S,a) \leftarrow \textbf{while}_T\ (\lambda(S,a).\ S{\neq}\{\})\ (\lambda(S,a).\ \textbf{do}\ \{ \\
\quad\ \ x{\leftarrow}\textbf{spec}\ x.\ x{\in}S; \\
\quad\ \ \textbf{return}\ (S{-}\{x\},a{+}x) \\
\quad \})\ (S,0); \\
\quad \textbf{return}\ a \\
\ \}
\end{array}
$$

Examples

Select item from (non-empty) set S : **spec** $x.\ x \in S$
Iterate until set is empty, sum up elements

sum_up S = **do** {
  $(S,a) \leftarrow$ **while**$_T$ $(\lambda(S,a).\ S \neq \{\})$ $(\lambda(S,a).$ **do** {
    $x \leftarrow$**spec** $x.\ x \in S$;
    return $(S - \{x\}, a + x)$
  }) $(S,0)$;
  return a
}

Only works for finite sets.

Examples

Check if $42 \in S$, S finite. Iterate over S.

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$is\_42in\ S = \textbf{do}\ \{$
$\quad (S,f) \leftarrow \textbf{while}_T\ (\lambda(S,f).\ S \neq \{\} \wedge \neg f)\ (\lambda(S,a).\ \textbf{do}\ \{$
$\qquad x \leftarrow \textbf{spec}\ x.\ x \in S;$
$\qquad \textbf{return}\ (S - \{x\}, x = 42)$
$\quad \})\ (S, False);$
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$$
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\quad\quad \textbf{return}\ (S - \{x\}, x = 42) \\
\quad \})\ (S, \mathit{False}); \\
\quad \textbf{return}\ f \\
\}
\end{array}
$$

We have emulated a *break* by Boolean flag f.

Edmonds-Karp Algorithm

```
let f = (λ_. 0);
(f,_) ← while_T
  (λ(f,brk). ¬brk)
  (λ(f,_). do {
    p ← select p. Graph.isShortestPath (residualGraph c f) s p t;
    case p of
      None ⇒ return (f,True)
    | Some p ⇒ do {
        let f = NFlow.augment_with_path c f p;
        return (f, False)
      }
  })
  (f,False);
return f
```

Comments

- Locale *Network* fixes flow network

```
locale Network
  fixes c :: nat × nat ⇒ 'capacity
    and s :: nat
    and t :: nat
  assumes Network c s t
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- **select** $p.\ \Phi\ p$: Nondeterministically select value that satisfies Φ
  - Returns *None* if there is no such term

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- Program m *refines* m', if results of m are also results of m'

  $\_ \leq FAIL$

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  - $m \leq RES\ (Collect\ \Phi)$ (notation: $m \leq (\textbf{spec}\ x.\ \Phi\ x)$)
- As Hoare-triple $\{P\}\ f\ \{Q\}$
  $P\ x \Longrightarrow f\ x \leq (\textbf{spec}\ r.\ Q\ r)$

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- Prove $\mathit{edmonds\_karp} \leq (\textbf{spec}\ f.\ \mathit{isMaxFlow}\ f)$
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- Use verification condition generator!

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- We will use wp here!

Standard Rules for *wp*

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    $\Longrightarrow wp\ (\textbf{while}\ b\ c\ s_0)\ (\lambda s.\ I\ s \wedge \neg b\ s)$
  - add consequence rule

    $\llbracket\ wf\ {<};\ I\ s_0;\ \bigwedge s.\ \llbracket b\ s;\ I\ s\rrbracket \Longrightarrow wp\ (c\ s)\ (\lambda s'.\ I\ s' \wedge s'{<}s);$
    $\quad \bigwedge s.\ I\ s \wedge \neg b\ s \Longrightarrow Q\ s\ \rrbracket$
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$$\Longrightarrow (\textbf{if}\ b\ \textbf{then}\ m_1\ \textbf{else}\ m_2) \leq (\textbf{spec}\ x.\ \Phi\ x)$$

$$[\![wf\ R;\ I\ s;\ \bigwedge s.\ [\![I\ s;\ b\ s]\!] \Longrightarrow f\ s \leq (\textbf{spec}\ s'.\ I\ s' \wedge (s', s) \in R);$$
$$\bigwedge s.\ [\![I\ s;\ \neg\ b\ s]\!] \Longrightarrow \Phi\ s]\!]$$
$$\Longrightarrow \textbf{while}_T\ b\ f\ s \leq (\textbf{spec}\ s.\ \Phi\ s)$$

...

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- Prove the generated VCs
  - Using Isabelle's standard proof methods

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  - Imply postcondition when loop terminates ($[\![I\ s;\ \neg b\ s]\!] \Longrightarrow \Phi\ s$)

Invariant Examples

Note: informal syntax!

- $a{=}0$; **while** $S{\neq}\{\}$ **do** $\{\ x{\leftarrow}$**spec** $x.\ x{\in}S;\ S{=}S{-}\{x\};\ a{=}a{+}x\ \}$
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- No subgoals left: lemma proved!

Simple_Invar_Demo.thy

Simple Invariants

`Edka_Abstract_Demo.thy`

Proving Correctness of Abstract Edmonds-Karp Algorithm

Conclusions (so far)

- Prove correctness of abstract algorithm first
  - can be modeled in nres-monad, shallowly embedded in HOL
- Proof by VCG + abstract theorems from background theory
  - VCG is almost automatic
  - Background theory can require considerable manual work

A Complete Example: State-Space Search

- Given directed edges E and a start node s, compute the set of reachable nodes

workset $W = \{s\}$; *visited set* $V = \{\}$

while $W \neq \{\}$ **do**
  remove some node u from W
  if $u \notin V$ **then**
    $V = V \cup \{u\}$
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- BFS, DFS, Best-First, ... are instances of this generic scheme!

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  - As start node is in V upon termination, we get $V \supseteq \mathit{reachable}$
- Termination: Only if set of reachable nodes is finite
  - $V \subseteq \mathit{reachable}$ increases, or V remains unchanged and $W \subseteq \mathit{reachable}$ decreases.

`Workset_Demo.thy`

Proving Correctness of State-Space Search

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- That's easy! But what if data representation changes?

Using Residual Graph

- Our current *edmonds_karp* computes residual graph in each iteration

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- Instead of flow, we can update residual graph
  - Upon termination: compute flow from residual graph

Refined Algorithm

Original:

```
let f = (λ_. 0);
(f,_) ← while_T
  (λ(f,brk). ¬brk)
  (λ(f,_). do {
    p ← select p. Graph.isShortestPath (residualGraph c f) s p t;
    case p of
      None ⇒ return (f,True)
    | Some p ⇒ do {
        let f = NFlow.augment_with_path c f p;
        return (f, False)
      }
  })
  (f,False);
return f
```

Refined Algorithm

Refined:

$\textbf{let } \mathit{cf} = c;$
$(\mathit{cf},\_) \leftarrow \textbf{while}_T$
$\quad (\lambda(\mathit{cf},\mathit{brk}).\ \neg \mathit{brk})$
$\quad (\lambda(\mathit{cf},\_).\ \textbf{do}\ \{$
$\quad\quad p \leftarrow \textbf{select}\ p.\ \mathit{Graph.isShortestPath\ cf\ s\ p\ t};$
$\quad\quad \textbf{case}\ p\ \textbf{of}$
$\quad\quad\quad \mathit{None} \Rightarrow \textbf{return}\ (\mathit{cf},\mathit{True})$
$\quad\quad \mid \mathit{Some}\ p \Rightarrow \textbf{do}\ \{$
$\quad\quad\quad\quad \textbf{let}\ \mathit{cf} = \mathit{Graph.augment\_cf\ cf}\ (\mathit{set}\ p)\ (\mathit{resCap\_cf\ cf\ p});$
$\quad\quad\quad\quad \textbf{return}\ (\mathit{cf},\ \mathit{False})$
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$\quad \})$
$\quad (\mathit{cf},\mathit{False});$
$\textbf{return}\ (\mathit{flow\_of\_cf\ cf})$

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Correctness

- How to prove this correct? w/o repeating abstract proof!
- Relate *edmonds_karp2* to *edmonds_karp*
- It's the same algorithm!
    - only flow has been exchanged by residual graph

Data Refinement

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- Show that operations on residual graph and flow are consistent

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- Show that operations on residual graph and flow are consistent
- Use structural rules to infer relation between programs

Relation between Residual Graph and Flow

- $\mathit{flow\_of\_cf}::(\mathit{nat} \times \mathit{nat} \Rightarrow {}'\mathit{capacity}) \Rightarrow \mathit{nat} \times \mathit{nat} \Rightarrow {}'\mathit{capacity}$
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    and *invariant* (*RGraph c s t*)
  - This pattern occurs frequently. Shortcut:
    $\mathit{cfi\_rel} \equiv \mathit{br}\ \mathit{flow\_of\_cf}\ (\mathit{RGraph}\ c\ s\ t)$

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  - Note the *isAugmentingPath* precondition!

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  - Note the *isAugmentingPath* precondition!
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- Can be expressed as refinement
  - $m_1 \le \Downarrow R\ m_2$, where

    $\Downarrow R\ FAIL = FAIL$

    $\Downarrow R\ (RES\ Y) = (\textbf{spec}\ x.\ \exists y{\in}Y.\ (x,\ y) \in R)$

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- Solution: assertions and congruence rules

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$$\llbracket b = b';\ \llbracket b;\ b' \rrbracket \Longrightarrow S1 \leq \Downarrow R\ S1';\ \llbracket \neg\ b;\ \neg\ b' \rrbracket \Longrightarrow S2 \leq \Downarrow R\ S2' \rrbracket$$
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  - equality/type-classes: $\mathit{List.member}::'a\ \mathit{list} \Rightarrow 'a \Rightarrow \mathit{bool}$. $(\mathit{List.member},\mathit{List.member}) \in ?$

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- Solution: Preconditions, generalization

Parametricity Examples

- First element of pair: $\mathit{fst}{::}'a \times {}'b \Rightarrow {}'a$
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Summary

- Apply refinement rules for combinators
  - Use some fallback rules to recover (slight) structural changes
  - Manually align bigger changes
- Guess unknown relations
  - By type, using natural relators for structured types
  - Manually
- Show refinements between operators
  - Insert enough assertions into abstract program to prove preconditions

Edka_Refine_Demo.thy

Edmonds-Karp on Residual Graphs

`Workset_Demo.thy`

Implementing Graph by Successor Function

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Imperative/HOL

- Model imperative program by state monad

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  datatype $('a,'h)\ M = M\ (run\colon \langle 'h \Rightarrow ('a \times 'h) \rangle)$

  return $x = M\ (\lambda s.\ (x,s))$
  $bind\ m_1\ m_2 = M\ (\lambda s.$ **let** $(x,s') = run\ m_1\ s\ in\ run\ (m_2\ x)\ s')$
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  - Haskell — has efficient heap monad

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- The Sepref Tool does exactly that!
  - And has large collection of readily available data structures

Imperative Refinement Basics

- Establish relation between imperative-program c and nres-program a

  $\mathit{hn\_refine}\ \Gamma\ c\ \Gamma'\ R\ a$

Γ	Heap content before execution
Γ'	Heap content after execution
R	Relation for result

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- Formally

  $\mathit{hn\_refine}\ \Gamma\ c\ \Gamma'\ R\ m \equiv$
  $\mathit{nofail}\ m \longrightarrow <\Gamma>\ c\ <\lambda r.\ \Gamma' * (\exists_A x.\ R\ x\ r * \uparrow(\mathbf{return}\ x \leq m))>_t$

  - where $<P>\ c\ <\lambda r.\ Q\ r>_t$ is Hoare-triple for Imperative/HOL programs

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- Refinement relations now also cover heap content.

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  - Arrays a_1 and a_2 do not overlap

Frame Rule

$$<P>\ c\ <Q> \Longrightarrow\ <P * F>\ c\ <\lambda r.\ Q\ r * F>_t$$

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  Rule for lookup in hashtable:
  $<\mathit{is\_hashmap}\ m\ ht>\ \mathit{hm\_lookup}\ k\ ht\ <\lambda r.\ \mathit{is\_hashmap}\ m\ ht * \uparrow(r = m\ k)>$

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  $<is\_hashmap\ m\ ht>\ hm\_lookup\ k\ ht\ <\lambda r.\ is\_hashmap\ m\ ht * \uparrow (r = m\ k)>$

  Of course, this still holds if also an array-list is on the heap

  $<is\_hashmap\ m_1\ ht_1 *$
  $\quad is\_array\_list\ l_2$
  $\quad\ al_2>\ hm\_lookup\ k$
  $\qquad\qquad ht_1\ <\lambda r.\ is\_hashmap\ m_1\ ht_1 * is\_array\_list\ l_2\ al_2 * \uparrow (r = m_1\ k)>_t$

Operation Refinement

$<$*is_hashmap m ht*$>$ *hm_lookup k ht* $<\lambda$*r. is_hashmap m ht* $* \uparrow (r = m\ k)>$

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Shortcut notations

$$
\begin{array}{lll}
(\mathit{hm\_lookup}, \lambda k\ m.\ m\ k) & \in \mathit{id}^k * \mathit{is\_hashmap}^k & \to \mathit{id} \\
(\mathit{hm\_update}, \lambda k\ v\ m.\ m(k{\mapsto}v)) & \in \mathit{id}^k * \mathit{id}^k * \mathit{is\_hashmap}^d & \to \mathit{is\_hashmap}
\end{array}
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k — keep

d — destroy

Synthesis of Program

Identify operations

do $\{$
  assert $(Q \neq \{\})$;
  $v \leftarrow$ **spec** $x.\ x \in Q \wedge (\forall y \in Q.\ x \leq y)$;
  case $m\ v$ **of**
    $None \Rightarrow$ **return** $(m(v \mapsto True))$
  $|\ Some\ \_ \Rightarrow$ **return** m
$\}$

Synthesis of Program

Identify operations

```
do {
  assert (Q≠{});
  v ← pq_get_min Q;
  case map_lookup v m of
    None ⇒ return (map_update v True m)
  | Some _ ⇒ return m
}
```

Synthesis of Program

Monadify (flatten expressions)

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\textbf{do}\ \{\\
\quad \textbf{assert}\ (Q \neq \{\});\\
\quad v \leftarrow \mathit{pq\_get\_min}\ Q;\\
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\qquad \mathit{None} \Rightarrow \textbf{return}\ (\mathit{map\_update}\ v\ \mathit{True}\ m)\\
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Initialize heap content

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\quad \textbf{assert}\ (Q \neq \{\}); \\
\quad v \leftarrow \mathit{pq\_get\_min}\ Q; \\
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\quad \textbf{case}\ t_1\ \textbf{of} \\
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Synthesis of Program

Symbolic forward execution

$$
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\quad v \leftarrow \mathit{pq\_get\_min}\ Q; \\
\quad t_1 \leftarrow \textbf{return } (\mathit{map\_lookup}\ v\ m) \\
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\}
\end{array}
$$

Synthesis of Program

Assert becomes no-op

$\textbf{do}\ \{$
$\quad (*\ \textbf{assert}\ (Q{\neq}\{\});\ *)$
$\quad <\ \mathit{minheap}\ \mathit{Qi}\ Q * \mathit{hashmap}\ \mathit{mi}\ m\ >\ |\ Q{\neq}[]$
$\quad v \leftarrow \mathit{pq\_get\_min}\ Q;$
$\quad t_1 \leftarrow \textbf{return}\ (\mathit{map\_lookup}\ v\ m)$
$\quad \textbf{case}\ t_1\ \textbf{of}$
$\qquad \mathit{None} \Rightarrow \textbf{do}\ \{t_2 \leftarrow \textbf{return}\ \mathit{True};\ \textbf{return}\ (\mathit{map\_update}\ v\ t_2\ m)\}$
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Synthesis of Program

$(\mathit{minheap\_get\_min}, \mathit{pq\_get\_min}) \in \mathit{minheap}^k \rightarrow \mathit{int}$

$\textbf{do}\ \{$
$\quad (*\ \textbf{assert}\ (Q{\neq}\{\});\ *)$
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Synthesis of Program

Result now also bound.

$\textbf{do}\ \{$
$\quad (*\ \textbf{assert}\ (Q{\neq}\{\});\ *)$
$\quad \mathit{vi} \leftarrow \mathit{minheap\_get\_min}\ \mathit{Qi};$
$\quad <\ \mathit{minheap}\ Q\ \mathit{Qi} * \mathit{hashmap}\ m\ \mathit{mi} * \mathit{int}\ v\ \mathit{vi}\ >\ |\ Q{\neq}[]$
$\quad t_1 \leftarrow \textbf{return}\ (\mathit{map\_lookup}\ v\ m)$
$\quad \textbf{case}\ t_1\ \textbf{of}$
$\qquad \mathit{None} \Rightarrow \textbf{do}\ \{t_2 \leftarrow \textbf{return}\ \mathit{True};\ \textbf{return}\ (\mathit{map\_update}\ v\ t_2\ m)\}$
$\quad |\ \mathit{Some}\ \_ \Rightarrow \textbf{return}\ m$
$\}$

Synthesis of Program

$(\textit{hm\_lookup}, \textit{map\_lookup}) \in \textit{int}^k * \textit{hashmap}^k \rightarrow (\textit{bool})\textit{option}$

$\textbf{do}\ \{$
$\quad (*\ \textbf{assert}\ (Q \neq \{\});\ *)$
$\quad vi \leftarrow \textit{minheap\_get\_min}\ Qi;$
$\quad <\ \textit{minheap}\ Q\ Qi * \textit{hashmap}\ m\ mi * \textit{int}\ v\ vi >\ |\ Q \neq []$
$\quad t_1 \leftarrow \textbf{return}\ (\textit{map\_lookup}\ v\ m)$
$\quad \textbf{case}\ t_1\ \textbf{of}$
$\qquad \textit{None} \Rightarrow \textbf{do}\ \{t_2 \leftarrow \textbf{return}\ \textit{True};\ \textbf{return}\ (\textit{map\_update}\ v\ t_2\ m)\}$
$\quad |\ \textit{Some}\ \_ \Rightarrow \textbf{return}\ m$
$\}$

Synthesis of Program

$$
\begin{array}{l}
\textbf{do}\ \{\\
\quad (*\ \textbf{assert}\ (Q \neq \{\});\ *)\\
\quad vi \leftarrow minheap\_get\_min\ Qi;\\
\quad ti_1 \leftarrow hm\_lookup\ vi\ mi;\\
\quad <\ minheap\ Q\ Qi * hashmap\ m\ mi * int\ v\ vi * (bool)option\ t_1\ ti_1\ >\ |\ Q \neq []\\
\quad \textbf{case}\ t_1\ \textbf{of}\\
\qquad None \Rightarrow \textbf{do}\ \{t_2 \leftarrow \textbf{return}\ True;\ \textbf{return}\ (map\_update\ v\ t_2\ m)\}\\
\quad |\ Some\ \_ \Rightarrow \textbf{return}\ m\\
\}
\end{array}
$$

Synthesis of Program

Split: Translate both branches separately

do $\{$
  (* **assert** ($Q \neq \{\}$); *)
  $vi \leftarrow minheap\_get\_min\ Qi$;
  $ti_1 \leftarrow hm\_lookup\ vi\ mi$;
  $<minheap\ Q\ Qi * hashmap\ m\ mi * int\ v\ vi * (bool) option\ t_1\ ti_1> \mid Q \neq []$
  case t_1 **of**
    $None \Rightarrow$ **do** $\{t_2 \leftarrow$ **return** $True$; **return** $(map\_update\ v\ t_2\ m)\}$
  $\mid Some\ \_ \Rightarrow$ **return** m
$\}$

Synthesis of Program

```
do {
  (* assert (Q≠{}); *)
  vi ← minheap_get_min Qi;
  ti₁ ← hm_lookup vi mi;
  case ti₁ of
    None ⇒ do {
      < minheap Q Qi * hashmap m mi * int v vi > | Q≠[], t₁=None
      t₂ ← return True;
      return (map_update v t₂ m)
      }
  | Some _ ⇒
      < minheap Q Qi * hashmap m mi * int v vi * bool t₃ ti₃ > | Q≠[],
t₁=Some t₃
      return m
}
```

Synthesis of Program

$(\textbf{return } \mathit{True}, \textbf{return } \mathit{True}) \in - \rightarrow \mathit{bool}$

do $\{$
  $(*$ **assert** $(Q \neq \{\}); *)$
  $\mathit{vi} \leftarrow \mathit{minheap\_get\_min}\ \mathit{Qi};$
  $\mathit{ti}_1 \leftarrow \mathit{hm\_lookup}\ \mathit{vi}\ \mathit{mi};$
  case ti_1 **of**
    $\mathit{None} \Rightarrow$ **do** $\{$
      $< \mathit{minheap}\ Q\ \mathit{Qi} * \mathit{hashmap}\ m\ \mathit{mi} * \mathit{int}\ v\ \mathit{vi} > \mid Q \neq [],\ t_1 = \mathit{None}$
      $t_2 \leftarrow$ **return** $\mathit{True};$
      return $(\mathit{map\_update}\ v\ t_2\ m)$
      $\}$
  $\mid \mathit{Some}\ \_ \Rightarrow$
      $< \mathit{minheap}\ Q\ \mathit{Qi} * \mathit{hashmap}\ m\ \mathit{mi} * \mathit{int}\ v\ \mathit{vi} * \mathit{bool}\ t_3\ \mathit{ti}_3 > \mid Q \neq [],$
$t_1 = \mathit{Some}\ t_3$
      return m
$\}$

Synthesis of Program

```
do {
  (* assert (Q≠{}); *)
  vi ← minheap_get_min Qi;
  ti₁ ← hm_lookup vi mi;
  case ti₁ of
    None ⇒ do {
      ti₂ ← return True;
     < minheap Q Qi ∗ hashmap m mi ∗ int v vi ∗ bool ti₂ t₂ > | Q≠[], t₁=None
      return (map_update v t₂ m)
      }
  | Some _ ⇒
      < minheap Q Qi ∗ hashmap m mi ∗ int v vi ∗ bool t₃ ti₃ > | Q≠[],
t₁=Some t₃
      return m
}
```

Synthesis of Program

$(\mathit{hm\_update},\mathit{map\_update}) \in \mathit{int}^k * \mathit{bool}^k * \mathit{hashmap}^d \to \mathit{hashmap}$

do $\{$
$(*$ **assert** $(Q \neq \{\}); *)$
$\mathit{vi} \leftarrow \mathit{minheap\_get\_min}\ \mathit{Qi};$
$\mathit{ti}_1 \leftarrow \mathit{hm\_lookup}\ \mathit{vi}\ \mathit{mi};$
case ti_1 **of**
$\mathit{None} \Rightarrow$ **do** $\{$
$\mathit{ti}_2 \leftarrow$ **return** $\mathit{True};$
$< \mathit{minheap}\ Q\ \mathit{Qi} * \mathit{hashmap}\ m\ \mathit{mi} * \mathit{int}\ v\ \mathit{vi} * \mathit{bool}\ \mathit{ti}_2\ t_2 > \mid Q \neq [], t_1 = \mathit{None}$
return $(\mathit{map\_update}\ v\ t_2\ m)$
$\}$
$\mid \mathit{Some}\ \_ \Rightarrow$
$< \mathit{minheap}\ Q\ \mathit{Qi} * \mathit{hashmap}\ m\ \mathit{mi} * \mathit{int}\ v\ \mathit{vi} * \mathit{bool}\ t_3\ \mathit{ti}_3 > \mid Q \neq [],$
$t_1 = \mathit{Some}\ t_3$
return m
$\}$

Destructive update, m_i no longer valid

```
do {
  (* assert (Q≠{}); *)
  vi ← minheap_get_min Qi;
  ti₁ ← hm_lookup vi mi;
  case ti₁ of
    None ⇒ do {
      ti₂ ← return True;
      hm_update vi ti₂ mi
      < minheap Q Qi * int v vi * bool ti₂ t₂ * hashmap @r @ri > | Q≠[],
t₁=None
      }
  | Some _ ⇒
      < minheap Q Qi * hashmap m mi * int v vi * bool t₃ ti₃ > | Q≠[],
t₁=Some t₃
      return m
}
```

Synthesis of Program

```
do {
  (* assert (Q≠{}); *)
  vi ← minheap_get_min Qi;
  ti₁ ← hm_lookup vi mi;
  case ti₁ of
    None ⇒ do {
      ti₂ ← return True;
      hm_update vi ti₂ mi
      < minheap Q Qi * int v vi * bool ti₂ t₂ * hashmap @r @ri > | Q≠[],
t₁=None
      }
  | Some _ ⇒
      < minheap Q Qi * hashmap m mi * int v vi * bool t₃ ti₃ > | Q≠[],
t₁=Some t₃
      return m
}
```

Pass on m_i as result. No aliasing, so m_i no longer valid!

$\mathbf{do}\ \{$
  $(*\ \mathbf{assert}\ (Q{\neq}\{\});\ *)$
  $vi \leftarrow minheap\_get\_min\ Qi;$
  $ti_1 \leftarrow hm\_lookup\ vi\ mi;$
  $\mathbf{case}\ ti_1\ \mathbf{of}$
    $None \Rightarrow \mathbf{do}\ \{$
      $ti_2 \leftarrow \mathbf{return}\ True;$
      $hm\_update\ vi\ ti_2\ mi$
      $<minheap\ Q\ Qi * int\ v\ vi * bool\ ti_2\ t_2 * hashmap\ @r\ @ri> \mid Q{\neq}[],$
$t_1{=}None$
      $\}$
  $\mid Some\ \_ \Rightarrow$
      $\mathbf{return}\ mi$
      $<minheap\ Q\ Qi * int\ v\ vi * bool\ t_3\ ti_3 * hashmap\ @r\ @ri> \mid Q{\neq}[],$
$t_1{=}Some\ t_3$
$\}$

Merge.

$\textbf{do}$ $\{$
  $(*$ $\textbf{assert}$ $(Q \neq \{\}); *)$
  $vi \leftarrow minheap\_get\_min\ Qi;$
  $ti_1 \leftarrow hm\_lookup\ vi\ mi;$
  $\textbf{case}\ ti_1\ \textbf{of}$
    $None \Rightarrow \textbf{do}\ \{$
      $ti_2 \leftarrow \textbf{return}\ True;$
      $hm\_update\ vi\ ti_2\ mi$
      $< minheap\ Q\ Qi * int\ v\ vi * bool\ ti_2\ t_2 * hashmap\ @r\ @ri > \mid Q \neq [],$
$t_1 = None$
      $\}$
  $\mid Some\ \_ \Rightarrow$
      $\textbf{return}\ mi$
      $< minheap\ Q\ Qi * int\ v\ vi * bool\ t_3\ ti_3 * hashmap\ @r\ @ri > \mid Q \neq [],$
$t_1 = Some\ t_3$
$\}$

Merge. ti_2 goes out of scope.

```
do {
  (* assert (Q≠{}); *)
  vi ← minheap_get_min Qi;
  ti₁ ← hm_lookup vi mi;
  case ti₁ of
    None ⇒ do {
      ti₂ ← return True;
      hm_update vi ti₂ mi
      }
  | Some _ ⇒
      return mi

  < minheap Q Qi ∗ int v vi ∗ hashmap @r @ri ∗ ∗ (bool)option t₁ ti₁ > | Q≠[]
}
```

Synthesis of Program

```
do {
  (* assert (Q≠{}); *)
  vi ← minheap_get_min Qi;
  ti₁ ← hm_lookup vi mi;
  case ti₁ of
    None ⇒ do {
      ti₂ ← return True;
      hm_update vi ti₂ mi
      }
  | Some _ ⇒
      return mi

  < minheap Q Qi * int v vi * hashmap @r @ri * * (bool)option t₁ ti₁ > | Q≠[]
}
```

Sepref_Demo.thy

Toy Example with Hashtable and Min-Heap

Workset_Demo_Impl.thy

Implementing the Workset Algorithm

`Edka_Impl_Demo.thy`

Implementation of Edmonds-Karp

Summary

- Select (imperative) data structures
- Synthesize Imperative/HOL program
- Generate ML/OCaml/Haskell/Scala program

Lecture Conclusions

- Prove algorithmic ideas on abstract level
  - No need to bother with implementation details
- Prove/reuse data-structures and sub-algorithms (independently)
  - Lot's of stuff has already been done: reuse, adapt, extend
- Use refinement to relate abstract with more concrete algorithms
  - Multiple steps. Step-size is a trade-off.
- Finally: Use Sepref to go to Imperative/HOL, generate code

Remarks on Learning Curve

- Should learn basic Isabelle first
  - E.g. Concrete Semantics book by Nipkow and Klein
- Then Refinement Framework (Look for tutorials in AFP)
  - Works quite smoothly
- Then Sepref Tool (Tutorial in AFP)
  - Can have quite subtle errors, needs some experience