SAT solver essentials, SAT modeling
Introduction

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VTSA School - Liege - 2021

Thanks to N. Szczepanski and L. Simon

Summary

- Introduction
- CDCL solvers
- SAT incremental
- Parallel solvers
- Encodings

```
http://cours.alfweb.net
vtsa / vtsa
```

The SAT problem

$$
\begin{array}{cccccc}
 & \overline{x_1} & \vee & \overline{x_2} & \vee & x_3 \\
\wedge & & & & & \overline{x_3} \\
\wedge & x_1 & \vee & x_2 & & \\
\wedge & & & x_2 & \vee & x_3
\end{array}
$$

- Variables : $x_1 \ldots x_3$, true or false
- Literals : x_1, $\overline{x_1}$ (or $\neg x_1$)
- Clauses : $\overline{x_1} \vee \overline{x_2} \vee x_3$
- CNF formula : conjunction of clauses

- SAT problem : is there an interpretation of variables that satisfy the CNF ?
- The canonical NP-Complete problem
- The easiest of hard problems.

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$$

x_1	x_2	x_3
F	F	F

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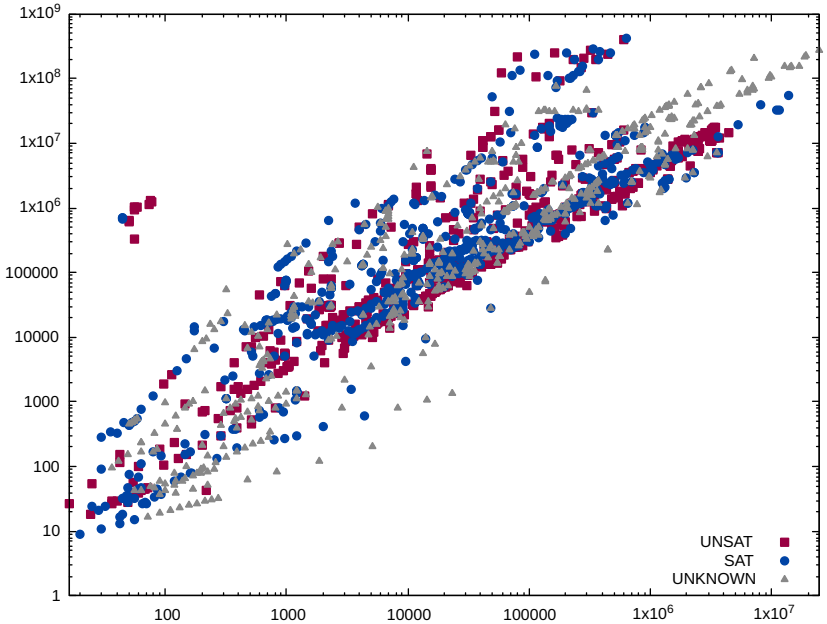
About size of formula

number of instructions	Time
$2^3 = 8$	immediate
$2^{37} \approx 80 \times 10^9$	1 second
$2^{56} \approx 8 \times 10^{16}$	$\approx$ 277 hours
$2^{60} \approx 10^{18}$	166 days
$2^{128} \approx 340 \times 10^{38}$	$\geq$ 3 billions of years



SAT Solving in Practice

- Before 2000, reduce a problem to SAT to show that it is intractable
- After 2000, reduce a problem to SAT to solve it efficiently
- SAT is a success story in computer science
- Many practical applications

Source J. Marques-Silva

Boolean Pythagorean Triple

SAT'16 result

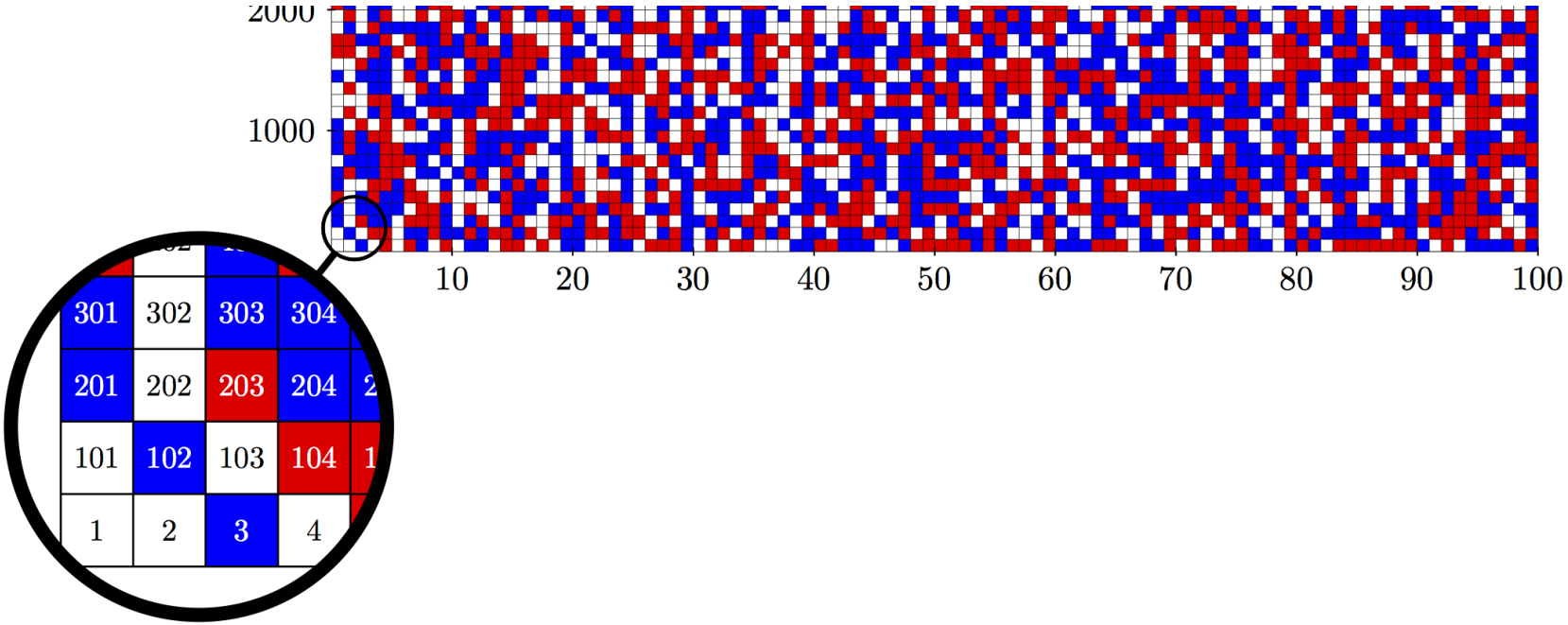
- Given a set $\{1, 2 \ldots n\}$, it is possible to partition this set in two subsets, such that no part contains a triple (a, b, c) such that $a^2 + b^2 = c^2$.
- For example, 3, 4 and 5 can not be in the same subset, since $3^2 + 4^2 = 5^2$

- The encoding is very simple
- One variable per integer
- The variable is set to true if it is in one subset
- The variable is set to false in the other one

```
p cnf 7820 18930
3 4 5 0
-3 -4 -5 0
5 12 13 0
-5 -12 -13 0
...
```

Boolean Pythagorean Triple

Using a SAT solver (aka glucose)



- A solution exists for $n = 7824$
- and...

Boolean Pythagorean Triple

- ...There is no solution for $n = 7825$, the resulted SAT problem is unsatisfiable
- A proof of size 200 terabytes ! !, The largest math proof ever !
- Of course, need of proof checkers ! !
- Is it really a proof ?

≪ If mathematicians work is understood to be a quest to increase human understanding of mathematics, rather than to accumu late an ever-larger collection of facts, a solution that rests on theory seems superior to a computer ticking off possibilities. ≫

[Nature].

- *ticking off possibilities ? ?*
- Much more sophisticated than that (symmetry breaking, search of cubes...)
- What about formal methods, SAT solving, Automated Theorem proving ?

Why studing CDCL solvers

- The logic behind SAT is simple
- A CDCL solver is not too difficult to implement (a version with 500 loc is available with the presentation)
- Many open source solvers are available : Minisat is the most famous one
- You can easily modify them and try new techniques/ideas
- You need to think carefully with data-structures (many variables/clauses)

But... many promising ideas are finally ineffective

Playing with SAT solvers is addictive

From DP to CDCL

1960 Davis and Putnam algorithm : based on resolution rule : forget one variable after the other

Resolution rule

$$(x \vee y \vee \neg z) \otimes_x (\neg x \vee y \vee t) = y \vee \neg z \vee t$$

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$$x_1 \vee x_4$$
$$\neg x_1 \vee x_4 \vee x_{14}$$
$$\neg x_1 \vee \neg x_3 \vee \neg x_8$$
$$x_1 \vee x_8 \vee x_{12}$$
$$x_1 \vee x_5 \vee \neg x_9$$
$$x_2 \vee x_{11}$$
$$\neg x_3 \vee \neg x_7 \vee x_{13}$$
$$\neg x_3 \vee \neg x_7 \vee \neg x_{13} \vee x_9$$
$$x_8 \vee \neg x_7 \vee \neg x_9$$

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$$x_4 \vee \neg x_3 \vee \neg x_8$$
$$x_8 \vee x_{12} \vee x_4 \vee x_{14}$$
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- Combinatorial explosion
- Used in pre-processing

some comparisons... even then

≪ The superiority of the present procedure (i.e. DP) over those previously available is indicated in part by the fact that a formula on which Gilmore's routine for the IBM 704 causes **the machine to compute for 21 minutes** without obtaining a result was worked successfully by **hand computation using the present method in 30 minutes** ≫

[Davis et Putnam 1960], page 202.

From DP to CDCL

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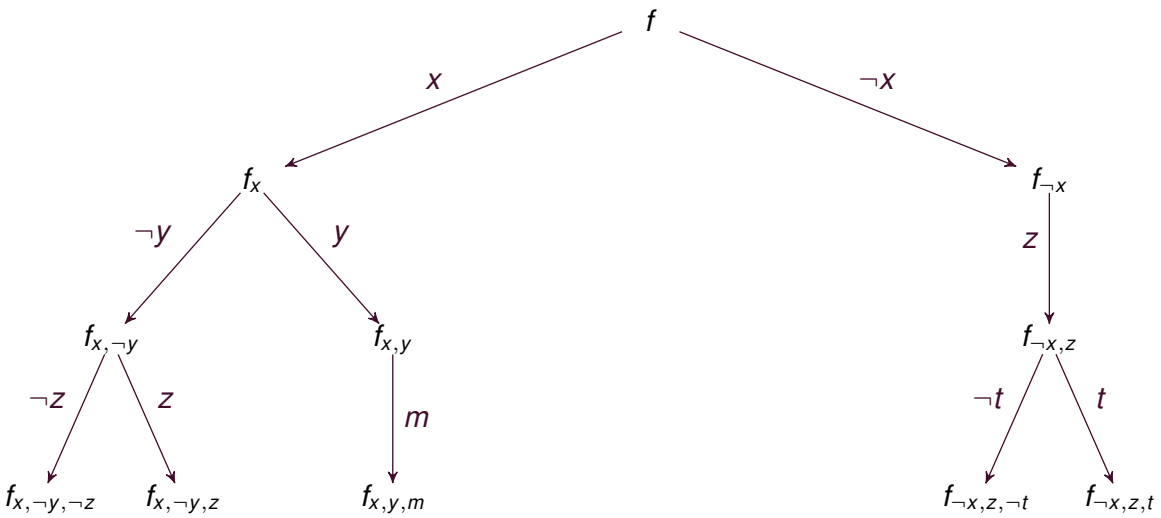
1962	Davis, Logeman Loveland algorithm : backtrack search tree (partial models)



- Decision variable is important
- Mistakes at the top of the search are dramatic
- A perfect branching scheme is NP-Hard (if the formula is SAT, I can make find a solution without backtrack)

From DP to CDCL

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1962 – 2001	Improve DPLL solvers : heuristics, look-ahead strategies...

From DP to CDCL

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1992	"*Planning as Satisfiability*". Kautz and Selman. ECAI
1999	"*Symbolic Model Checking Without BDD*". Biere, Cimatti, Clarke and Zhu. TACAS

- Encoding of application problems in SAT

From DP to CDCL

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1996	"*GRASP - a new search algorithm for satisfiability*".Sakallah and Marques-Silva. ICCAD
2001	"*Chaff : Engineering an Efficient SAT Solver*". Moskewicz, Madigan, Zhao, Zhang and Malik. DAC
2003	"*An extensible SAT solver*". Een and Sorensson. SAT

- Learning + dedicated data structures for managing huge formulas

Before 2000

- Look Ahead solvers

  To know where to go, you need to know where you are

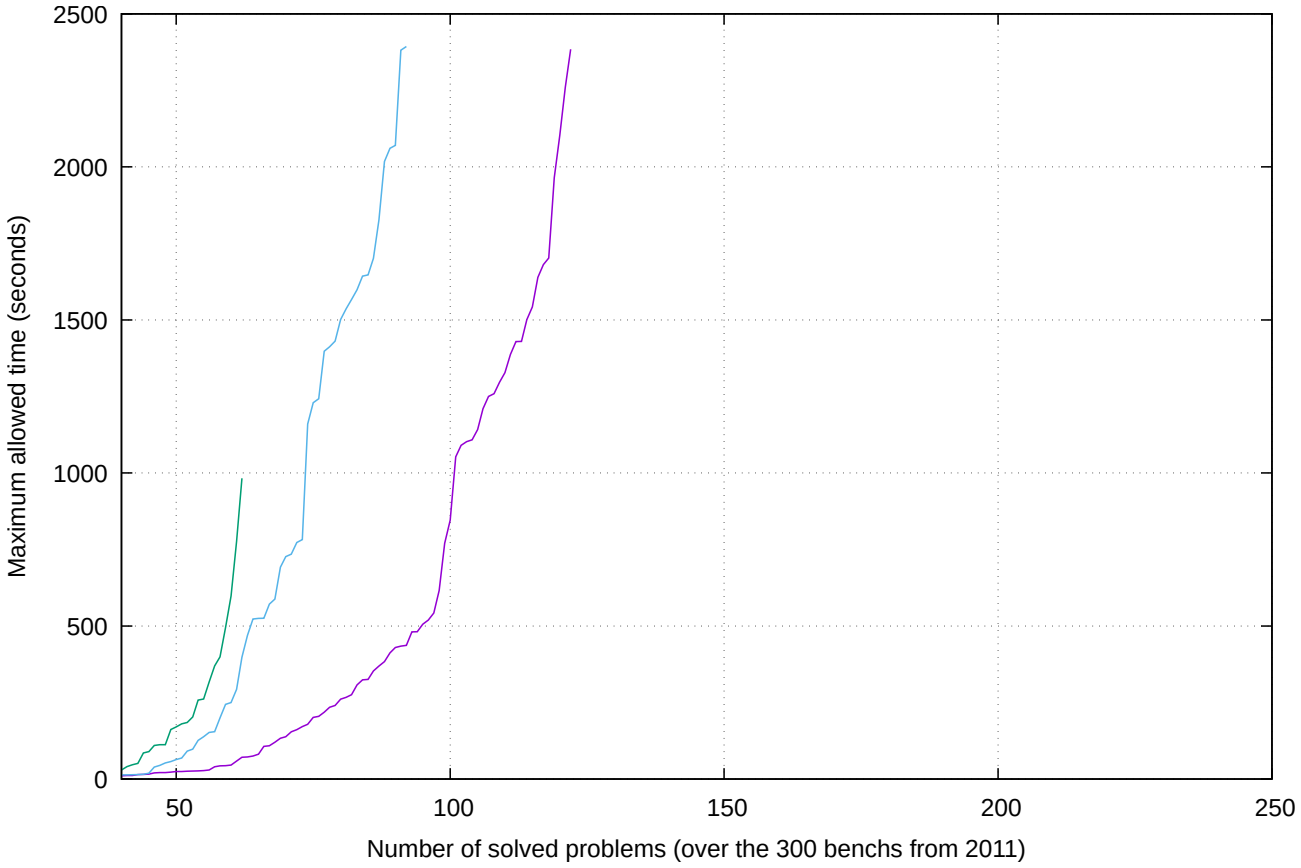
  - Search tree
  - Ideas/Techniques that aim to reduce/balance the search tree.
  - Heuristics, failed literals
  - Explanation of performances is (relatively) simple

- Lookback solvers (CDCL)

  You do not know where you are, but you know where to go

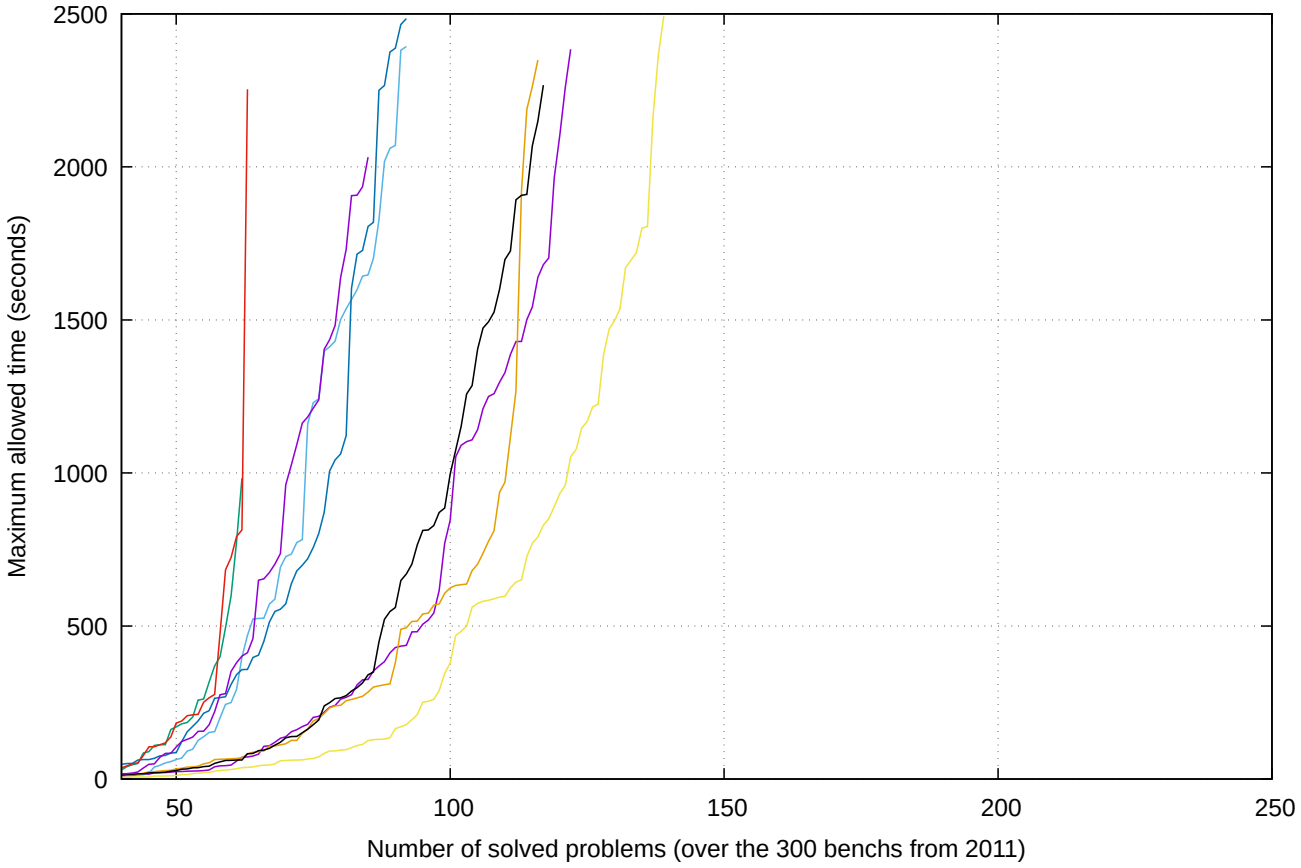
  - Many variables, many clauses
  - Restarts
  - Explanation of performances is quite hard

Performances after 2001



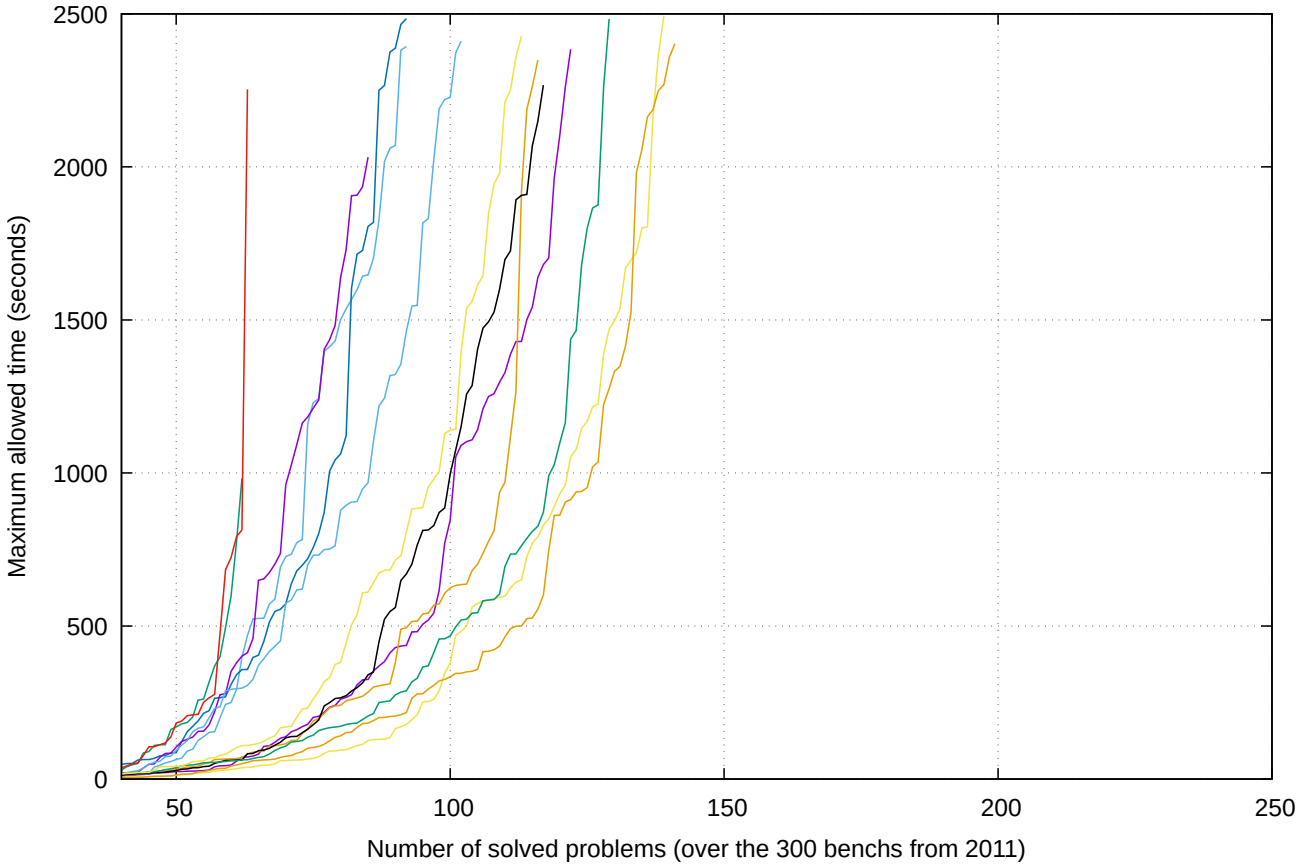
2002

Performances after 2001



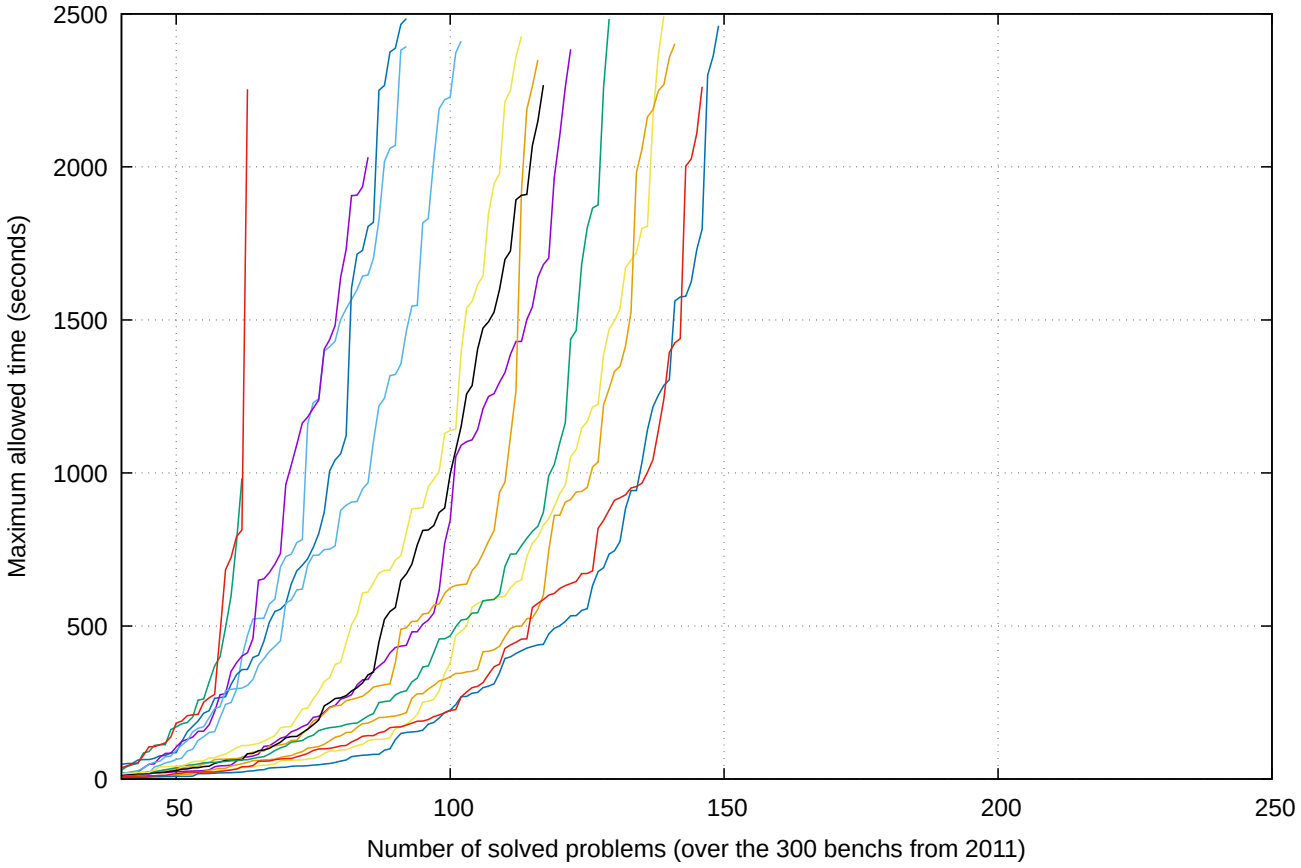
2003

Performances after 2001



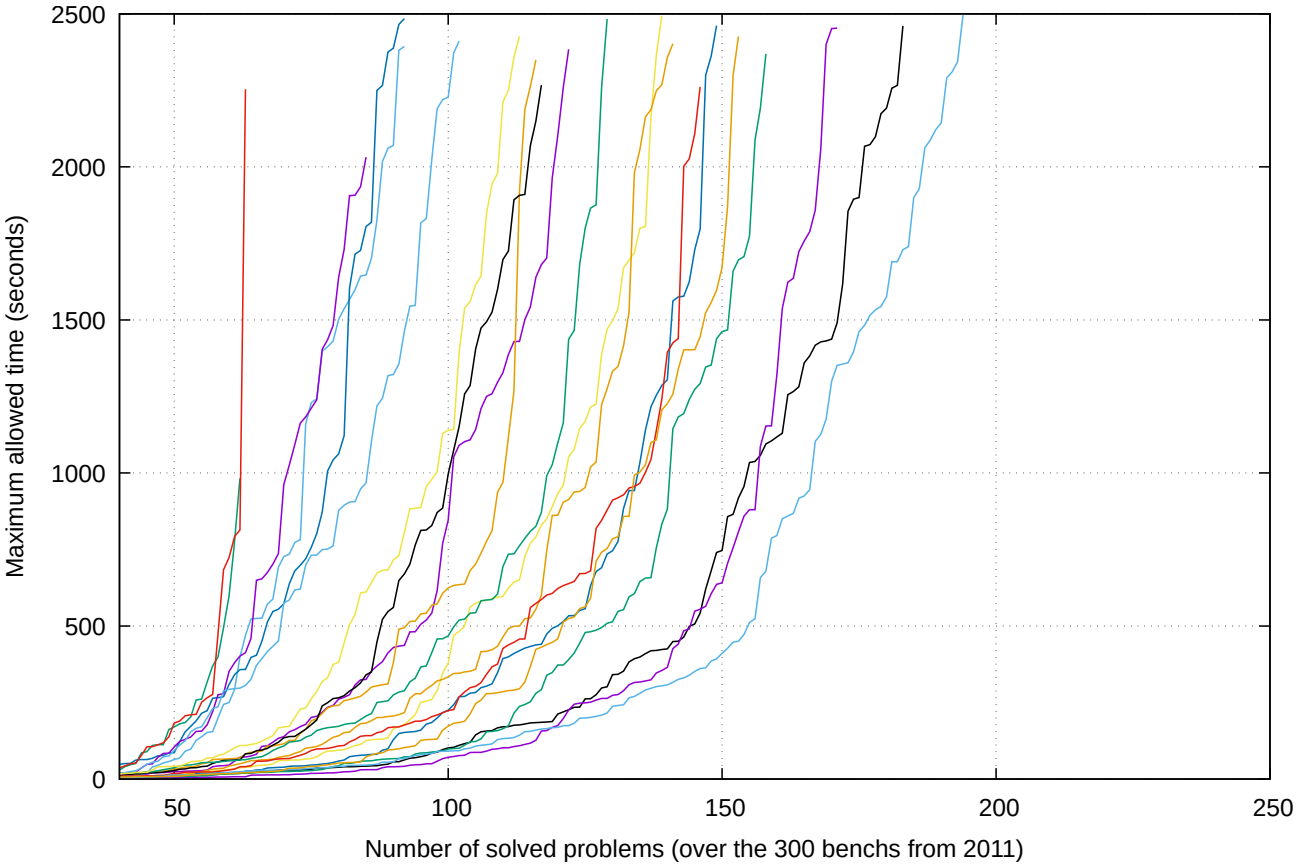
2005

Performances after 2001



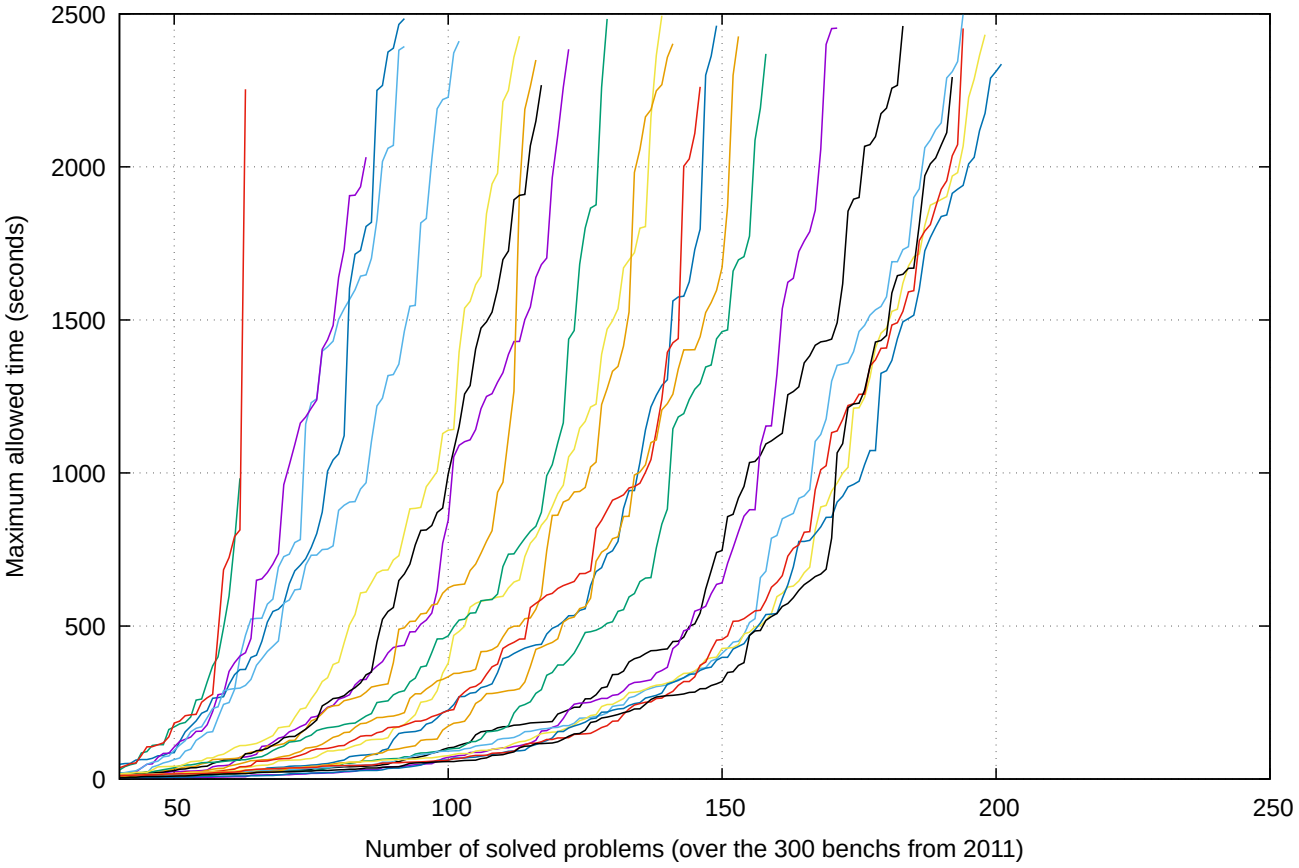
2007

Performances after 2001



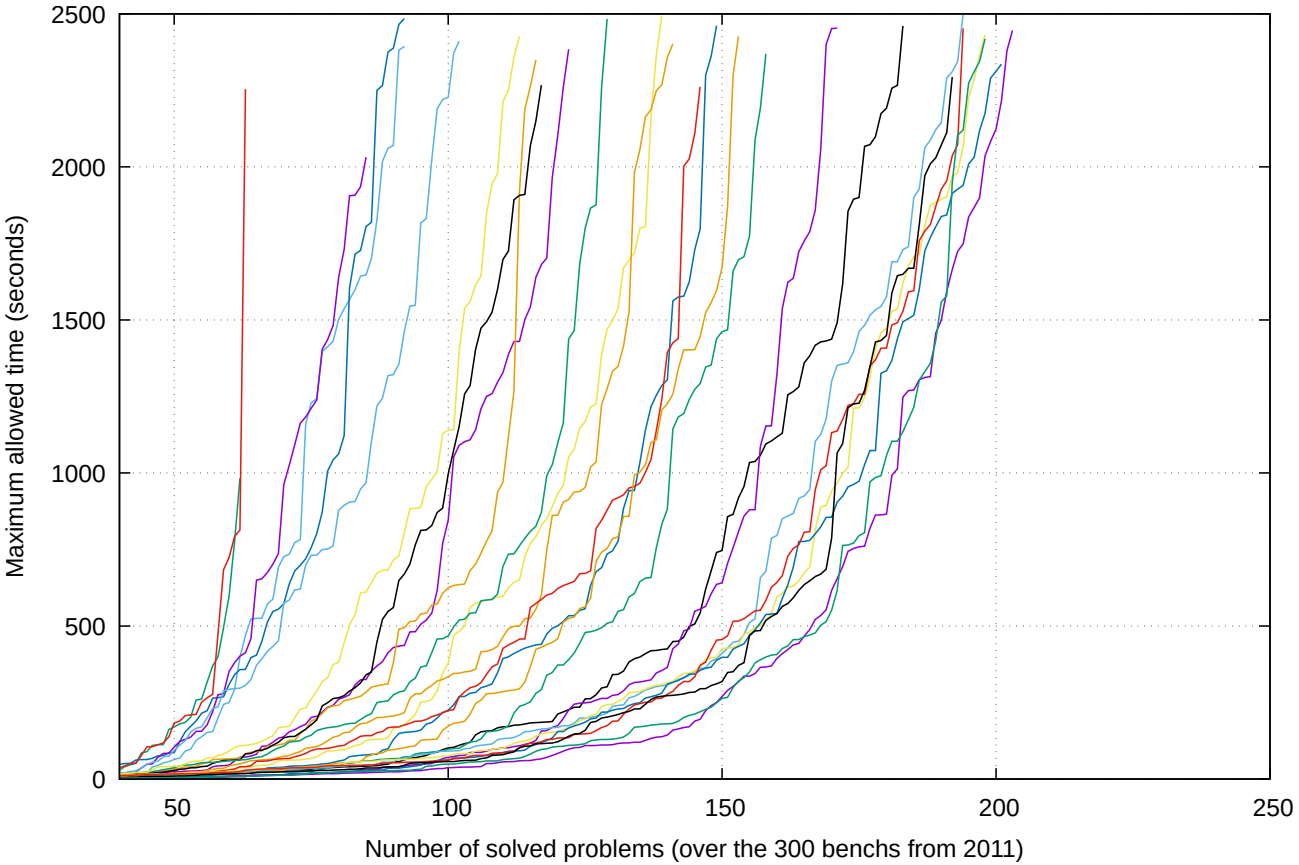
2009

Performances after 2001



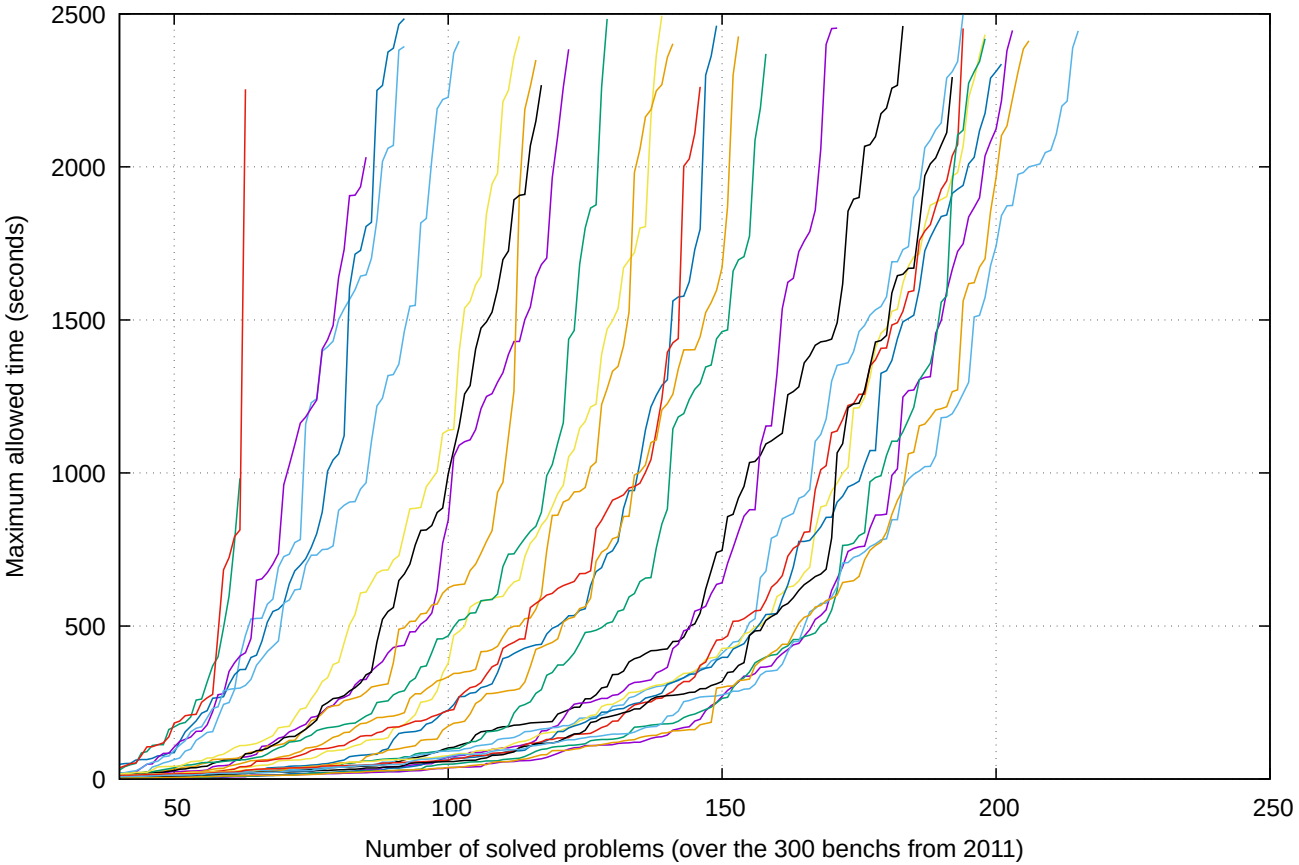
2011

Performances after 2001



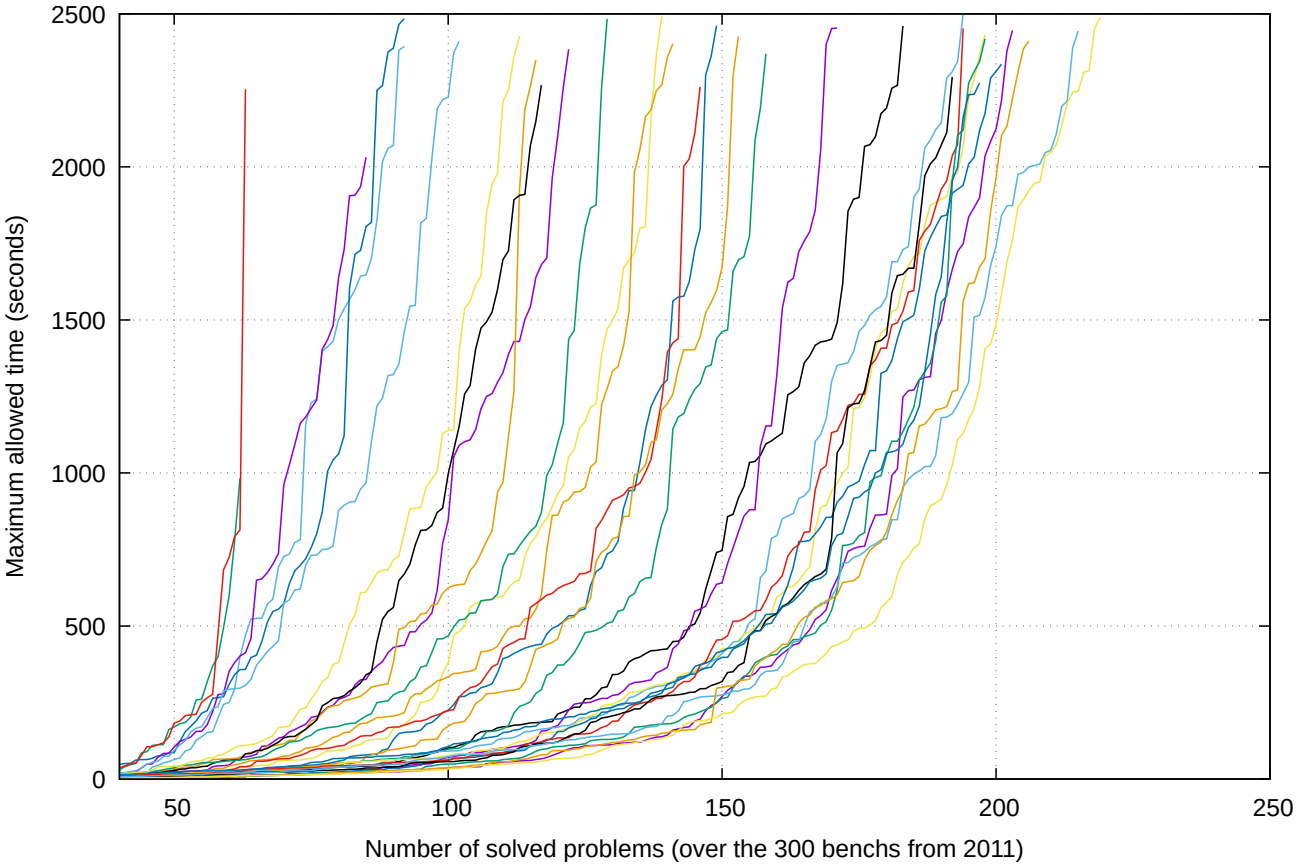
2014

Performances after 2001



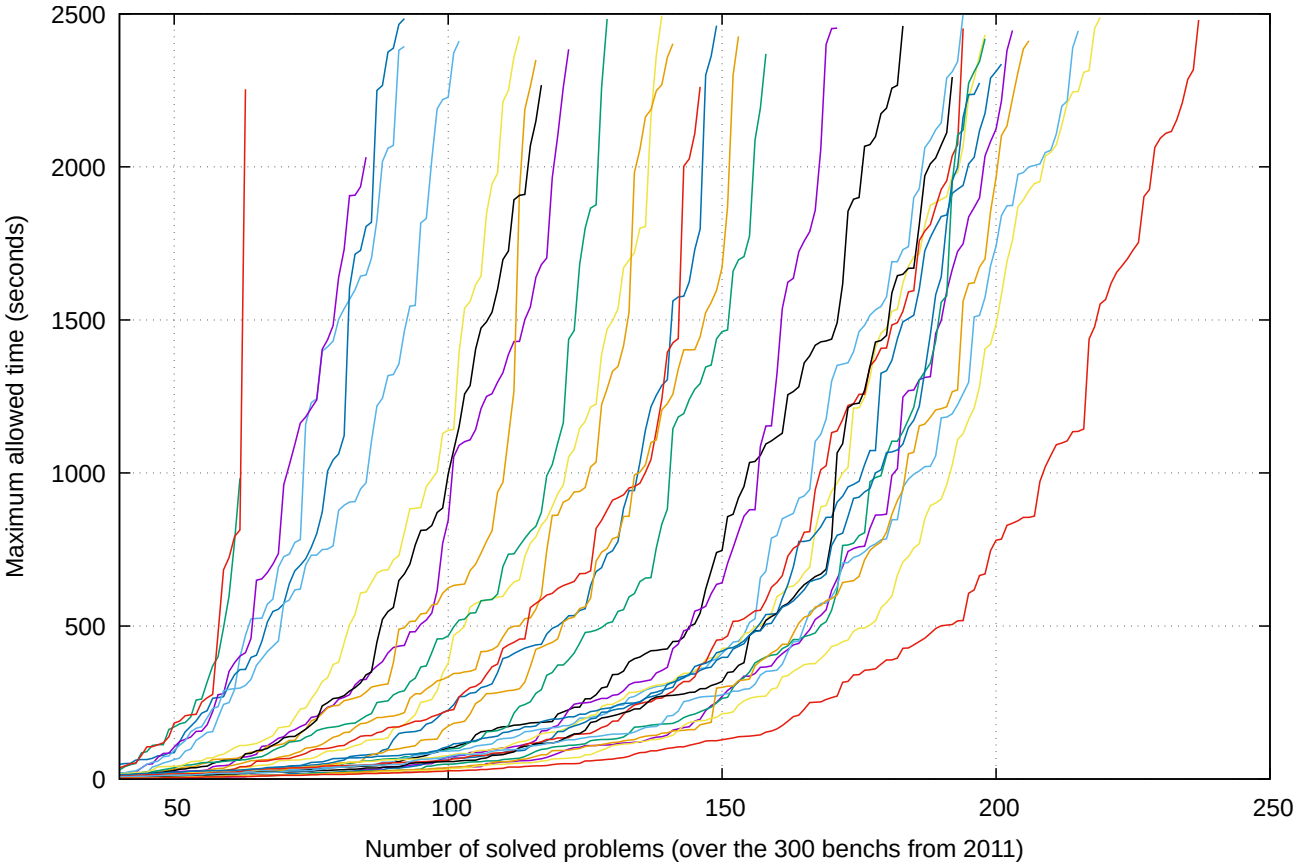
2016

Performances after 2001



2018

Performances after 2001



2020

Some SAT solvers

The precursors

Grasp (1996) Learning
Relsat (1996) Learning
Zchaff (2000) Lazy data structures

Minisat based

Minisat (2003), the original one
SAT4J (2004), in Java
RSat (2007), phase saving
Glucose (2009), quality of learnt clauses
Cryptominisat (2009), XOR reasoning
MapleXXXX (2016...), based on glucose. Add LRB heuristic and other features

Armin Biere solvers

Precosat (2009)
lingeling (2010)
cadical (2019)
Kissat (2020)

SAT Heritage

- `https://github.com/sat-heritage/docker-images`
- Software heritage for the SAT community
- A collection of SAT solvers that participate to some SAT competitions (632)
- Local search, parallel, CDCL, DPLL...
- You can search, download, build, launch them
- Based on Docker

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Exercise
Pythagorean triples

A word on 3-SAT

- k-SAT : Exactly k literals per clause
- 2-SAT is polynomial
- 3-SAT is NP-Complete